Modeling and Simulation of Low Voltage Arcs

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Technische Universiteit Delft, op gezag van de Rector Magnificus Prof. ir. K.C.A.M. Luyben, voorzitter van het College voor Promoties, in het openbaar te verdedigen

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To our families
Summary

Current interruption is the core technology in low voltage circuit breakers and it is characterized by the presence of an electric arc. This thesis is an attempt to provide a contribution to the modeling and the comprehension of such a complex phenomenon, and possibly to indicate a predictive approach for its computational simulation.

An experimental campaign is carried out, studying the behavior of real industrial breakers under short-circuit tests. The testing network is modeled in the framework of a lumped parameter approach, and suitable tests have been carried out to identify stray capacitances. Electrical measures are performed, including a special current sensor for the post-arc phase. A suitable signal regularization technique is defined, in order to filter noise out. Optical measures, based on fiber optics, are also used [90] and allow studying the motion of the arc and, particularly, of its roots. A set of performance evaluators are defined, which are observed to be correlated with the quality of the interruption. For each evaluator, a threshold value discriminates in between successful interruptions and failures. We got experimental evidence of the correlation of the current zero region with the final outcome of the interruption [9]. On the other hand, the arc behavior clearly appears to be extremely scattered, even in nominally identical conditions.

A theoretical description of the electric arc is outlined, based on plasma physics. The chemical composition of air plasma is first described. Then the motion of interacting charged particles in an electric field is used to introduce the mechanism of energy transfer through collisions, especially between particle species of similar mass, and the mechanism of energy gain from the electric field, especially for particle species of lighter mass, i.e., electrons. The balance of the two mechanism determines the fulfillment of the local thermal equilibrium. The thin arc root regions and the cold plasma condition when the arc is close to
extinction in the current zero region are supposed to require non equilibrium theory for a reliable modeling. Based on this, we propose \[8\] a black box arc model for low voltage circuit breakers, consisting of the classical Schwarz model for the high current regime and introducing a correction in the low current regime, so to account for non equilibrium physics. A general model parameter identification method is proposed \[8\], based on solving a constrained optimization problem by suitably coupling gradient moves with heuristic search methods. A very good agreement of the proposed model for low voltage arcs is found with experimental data \[8\], introducing significant improvements to equilibrium based models.

The large scattering in the arc behavior prevents the black box approach from being predictive in different conditions than those used for its identification. A computational, multiphysical approach is introduced and suggested for a predictive simulation. The magnetohydrodynamics (MHD) description is proposed, supplemented by suitable models for non equilibrium dominated features, such as the arc roots, that could not be resolved. The scope of the model is the macroscopic scale of the arc dynamics as a conducting, compressible, viscous fluid, driven by electromagnetic forces and pressure gradients. Radiative heat transfer in a participating medium accounts for the primary heat dissipation and redistribution mechanism. In the typical conditions found in low voltage circuit breakers, the magnetic Reynolds number is seen to be low and a weakly coupled approach is thus proposed for solving Navier-Stokes and Maxwell equations independently. The Galerkin method is used to discretize in space the governing PDE, adopting in particular the finite volume method for fluid dynamics and radiative heat transfer, the node based finite element method for electrostatics and finally the curl conform, edge finite element method for magnetostatics. Data passing is accomplished by means of a distance weighted interpolation scheme.
Samenvatting

Bij het ontwerpen van laagspanningschakelaars staat stroomonderbreking door middel van lichtbogen centraal. Dit proefschrift beoogt een bijdrage te leveren aan het modeleren en het begrijpen van de complexe processen die hierbij optreden. Daarbij is het oogmerk geweest rekenkundige modellen te ontwikkelen die bij het ontwerp proces gebruikt kunnen worden.

In het kortsluitlaboratorium is een onderzoeksprogramma uitgevoerd om het gedrag van industriële prototypes tijdens kortsluiting te bestuderen. Het beproevingscircuit is gemodelleerd met discrete componenten en de strooi inductiviteiten en capaciteiten zijn proefondervindelijk bepaald. De stroom rond het moment van afschakelen is met een speciaal voor dit doel ontwikkelde stroom sensor gemeten. De ruis is met een geavanceerde filter techniek uit het meetsignaal verwijderd. Met een optische meetopstelling, gebaseerd op glasvezel techniek, is het gedrag van de lichtboog tijdens het afschakelen bestudeerd.

Er zijn een aantal indicatoren gedefinieerd die stroomonderbreking kwantificeren. Voor elke indicator is een drempelwaarde vastgesteld die de grens tussen een succesvolle onderbreking en een niet-succesvolle onderbreking aangeeft. Op experimentele wijze is er een correlatie bepaald tussen de processen rond stroomnul en het succes van de stroomonderbreking. Het gedrag van de lichtboog vertoont onder identieke beproevingscondities een grote spreiding.

Gebaseerd op de fundamenten van de plasmafysica wordt het gedrag van de lichtboog beschreven. Eerst wordt de scheikundige samenstelling van het plasma van een in lucht brandende boog beschreven. Vervolgens wordt doormiddel van de interactie van geladen deeltjes met het elektrisch veld het mechanisme van energie overdracht door botsingen, in het bijzonder bij deeltjes van gelijke massa, en het mechanisme van energie overdracht vanuit het elektrisch veld aangegeven.
trisch veld, in het bijzonder voor deeltjes met een geringe massa zoals elektronen, beschreven. De balans tussen beide mechanismen bepaalt het tot stand komen van een lokaal thermisch evenwicht. De voetpunten van de boog en het koude plasma ten tijde van de stroomonderbreking bij een stroom nul doorgang, vereisen voor een betrouwbare model echter een beschouwing van de niet-evenwicht toestand. Op basis van deze constatering komen wij tot een black-box lichtboogmodel voor laagspanningschakelaars, gebaseerd op het klassieke Schwartz model voor het hoogstroom gebied, met als toevoeging een aanpassing voor het lage stromen interval, om op deze wijze de niet-evenwicht situatie daarin te verdisconteren. Er wordt een algemene parameter identificatie methode gecreëerd die is gebaseerd op het oplossen van een beperkt optimalisatie probleem in combinatie met heuristische zoek methoden. Er is een goede overeenkomst gevonden tussen het ontwikkeld theorethische concept en de experimentele data. Dit is een duidelijke verbetering ten opzichte van de op thermisch evenwicht gebaseerde modellen.

De grote spreiding in het gedrag van de lichtboog maakt het toepassen van de black-box benadering bij het voorspellen van het afschakelgedrag onder andere beproevingscondities lastig. Hiervoor wordt een multifysische benadering gekozen. De magneto hydrodynamische (MHD) beschrijving van het boog gedrag wordt daarbij als uitgangspunt genomen, aangevuld met modellen voor niet-evenwichtssituaties, zoals die optreden bij de voetpunten van de licht boog.

Kenmerkend voor het ontwikkelde model zijn de macroscopische schaal waarop de geleiding van de boog, de samendrukbaarheid van het plasma, het viskeuze gedrag van de boog onder invloed van elektromagnetische krachten en drukverschillen, een rol spelen. De primaire energie dissipatie wordt veroorzaakt door straling. Het Reynolds-getal is voor laagspanning schakelaars klein en dien overeenkomstig kan er bij het oplossen van de Navier-Stokes en Maxwell vergelijkingen gebruik gemaakt worden van een zwakke koppeling. De partiële differentiaal vergelijkingen worden opgelost met de methode van Galerkin, gebruik makende van de eindige elementen methode voor de vloeistof mechanica en de energie overdracht voor straling.
La capacità di interrompere la corrente è il dato tecnologico fondamentale di un interruttore di bassa tensione ed è caratterizzata dalla presenza dell’arco elettrico. Questa tesi si propone di fornire un contributo alla comprensione e alla modellazione di questo complesso fenomeno, e possibilmente di indicare un approccio computazionale e predittivo per la sua simulazione.

Il comportamento di veri interruttori, prodotti a livello industriale, è stato analizzato per mezzo di una vasta campagna sperimentale di prove di corto circuito. La rete di prova è stata accuratamente descritta per mezzo di un modello a parametri concentrati, utilizzando prove specifiche per identificare le capacità parassite. I test sono stati oggetto di misure elettriche standard e, in aggiunta, si è utilizzato un sensore speciale per misurare con accuratezza la debole corrente che si registra in fase post arco. Si è definita un’apposita tecnica di regolarizzazione del segnale, allo scopo di rimuovere il rumore. Per mezzo di misure con fibre ottiche è stato possibile filmare la dinamica dell’arco, ed in particolare delle sue radici. Abbiamo definito un numero di indicatori di merito che l’evidenza sperimentale mostra essere in correlazione con il successo nell’interrompere la corrente. Per ciascuno di essi, un valore di soglia separa le interruzioni riuscite dai fallimenti. Abbiamo ottenuto evidenza sperimentale della correlazione tra alcuni dati disponibili nella regione prossima allo zero di corrente e l’esito finale dell’interruzione [9]. D’altra parte, il comportamento dell’arco appare essere estremamente variabile e poco ripetibile, perfino in condizioni nominalmente identiche.

Proponiamo una sintetica introduzione teorica all’arco elettrico quale argomento di fisica dei plasmi. Descriviamo inizialmente la composizione chimica della miscela che si trova in un plasma di aria. Quindi ci serviamo della descrizione al livello di un sistema di particelle cariche e interagenti elettromagneticamente all’interno di un campo elettrico per introdurre il meccanismo
di redistribuzione energetica per mezzo di collisioni, specialmente per specie chimiche con massa simile, ed il meccanismo di cessione energetica dal campo elettrico alle particelle, specialmente quelle più leggere, cioè gli elettroni. Il bilancio dei due meccanismi determina il soddisfacimento della condizione di equilibrio termico locale. Si ritiene che la teoria del non equilibrio sia necessaria per descrivere realistamente la fisica degli strati fini ai piedi dell’arco e la condizione di plasma freddo, quando l’arco è prossimo alla sua estinzione nella regione dello zero di corrente. Basandoci su questo schema teorico, proponiamo un modello black box per gli interruttori di bassa tensione, che coincide con il classico modello di Schwarz per il regime di alta corrente e in cui si introduce una correzione nel regime di bassa corrente, così da tenere in conto della fisica del non equilibrio. Proponiamo un metodo generale per l’identificazione parametrica di modelli black box di arco, che si riconduce alla soluzione di un problema di ottimizzazione vincolata per mezzo di un approccio a gradiente opportunamente accoppiato a metodi di ricerca diretta di natura euristica. Si ottiene un ottimo accordo tra i dati sperimentali e il modello proposto, introducendo miglioramenti significativi rispetto ai modelli che non considerano i fenomeni di non equilibrio.

L’ampia variabilità e scarsa ripetibilità del comportamento dell’arco impedisce ai modelli black box di essere predittivi in condizioni diverse da quelle usate per identificarne i parametri. Pertanto si introduce e si consiglia un approccio di natura multifisica per fini predittivi. Alla classica base magnetoidrodinamica occorre aggiungere opportuni modelli che descrivano fenomeni fisici dominati da effetti di non equilibrio, come le radici dell’arco, che non potrebbero altrimenti essere risolti. L’obiettivo è di cogliere la scala macroscopica della dinamica dell’arco, visto come un fluido comprimibile viscoso e conduttivo (in ragione della temperatura locale), su cui agiscono forze elettromagnetiche e gradienti di pressione. Il meccanismo dominante di dissipazione energetica e redistribuzione termica è l’irraggiamento con mezzo partecipante. Si mostra che, nelle condizioni tipiche che ricorrono nel plasma d’arco in bassa tensione, il numero di Reynolds magnetico è basso, cosa che permette l’accoppiamento debole tra le equazioni di Navier-Stokes e di Maxwell nella soluzione del problema magnetoidrodinamico. Le equazioni governanti sono approssimate numericamente per mezzo del metodo di Galerkin, utilizzando in particolare il metodo dei volumi finiti per la fluidodinamica e l’irraggiamento con mezzo partecipante, gli elementi finiti nella classica versione nodale per l’elettrostatica ed infine gli elementi finiti di tipo edge per la magnetostatica. Il passaggio dati avviene per mezzo di uno schema di interpolazione pesata con la distanza dei punti campionati dal punto interpolando.
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Foreword

The present joint thesis rests on the research activity carried out together by Andrea Balestrero and Luca Ghezzi. Each candidate is responsible for the arguments developed throughout the thesis as hereafter detailed. The “Joint development” label addresses those topics developed with a common and virtually indistinguishable effort by both of the two authors.

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Chapter 1

Introduction

1.1 Low Voltage Circuit Breakers

The present thesis is focused onto the interruption capability of low voltage (LV) circuit breakers (CB), which is inescapably characterized by the presence of electric arcs. According to a wider and abstract description, circuit breakers are electro-mechanical devices, to be inserted into electric networks, whose function is four-fold. Precisely, a low voltage circuit breaker is required to:

1. *Make the current*, that is, to close an electric circuit (which could be a portion of or the whole electric network at hand) under an external command and from an initial open state with no electric current flowing;

2. *Carry the current*, that is, when closed, to withstand, under nominal working conditions and for an indefinitely long time, the flow of an electric current falling into a range prescribed by the Standards;

3. *Break the current*, that is, to open the circuit under an external command or after the detection of a dangerous condition, as prescribed by the Standards, and timely switch to a state with no electrical current flowing;

4. *Isolate a part of the circuit from the supply*, that is, when open, to withstand a voltage under conditions prescribed by the Standards, without letting an electric current flow into the circuit.
CHAPTER 1. INTRODUCTION

The main purpose is obviously to break and electrically disconnect a portion of the network, and particularly the devices located therein, together with the humans possibly in contact with them, so to prevent a harmful condition from occurring. All other requirements are necessary conditions to coexist with an electric network. The above mentioned harmful conditions are three-fold, namely:

1. A short-circuit, that is, a current peak, typically very highly exceeding nominal working conditions and possibly very limited in time, as prescribed by the Standards;

2. An over-current\(^1\), that is, a current slightly exceeding nominal working conditions, but for possibly a long time, as prescribed by the Standards;

3. A residual current, that is, a current flowing from phase to phase or from phase to ground, possibly through the body of a human in direct or indirect contact with the network, and usually ranging very limited values with reference to nominal working conditions, as prescribed by the Standards.

Since the most dangerous arcing conditions are found in short-circuits, this thesis is mostly focused on the first category, especially with reference to its experimental content, even though some of the findings, especially with reference to the theoretical basis and simulation approaches, are also applicable to a wider context.

1.1.1 The Electric Arc Plasma

The main task and characterizing feature which is expected from a circuit breaker is the capability to timely commutate from the conductive state to the insulating state, and vice versa. An ideal breaker would commutate instantaneously, with a discontinuity in electric current time history and thus a singularity in its time derivative. This is clearly not admissible in any real circuit, where some inductance (at least a small stray inductance) is always present. As a matter of fact, in any real circuit breaker the commutation is (hopefully) fast but not instantaneous and, as soon as the electric contacts are detached and moved apart, then an electric arc is drawn, providing the continuity of electric current time history.

\(^1\)The term “over-current” is sometimes used to refer to any current exceeding the nominal value, without any further specification and thus including short-circuits.
1.1. LOW VOLTAGE CIRCUIT BREAKERS

The arc is an electric discharge which takes place in the hot and ionized, and thus conducting, air gap in between the detached electric contacts. During an interruption, the heating of the surrounding air is initiated as soon as contact pressure begins diminishing in still closed contacts, due to the incipient opening maneuver. The contact area, which is determined by the contact pressure and the stiffness and geometry of the contacts, is progressively reduced, with a correspondent increment of the electric resistance and Joule heating. The heat thus produced first dissociates and then ionizes molecules and atoms, bringing air into the state of a plasma.

The term plasma was coined by Irving Langmuir, who pioneered this field of research, from the greek word πλασµα, meaning a moldable substance. Also termed the fourth state of matter, after the solid, liquid and gaseous states, plasma is intrinsically an ionized gas, with electrons removed from molecules or atoms being charge carriers that guarantee a discrete conductance, which could become superior to that of metals in the case of the very rarefied and hot plasmas used in nuclear fusion research. In the case of high density, that is, approximately around normal pressure conditions, and typical temperatures of low voltage circuit breaker arc plasma, that is, approximately in the $10000 - 20000 \ K$ range, the electrical conductivity can be as high as some $10^4 \ S/m$, that is, 3 orders of magnitude lower than that of metals, but many orders of magnitude higher than insulating, cold air.

Owing to the highly nonlinear temperature dependence of its electrical conductivity, the arc is a physical phenomenon allowing a very fast commutation from the conductive to the insulating state. On the other hand, the arc introduces a series of very negative consequences into the circuit breaker. The ohmic power dissipated into heat in its inside produces material ablation both from the electrical contacts and from the surrounding plastic enclosures, so that the breaker gets progressively degraded after each operation. Moreover, the pressure build up, induced by thermal power, could also be disruptive and lead to the explosion of the circuit breaker. Probably the most negative drawback of the low voltage electric arc is its scarce repeatability, even under seemingly similar conditions, which is due to its property of being an intrinsically unstable phenomenon.

It is apparent that the modeling of any circuit breaker, and particularly low voltage ones, is intrinsically the modeling of an arc plasma, whose dynamical evolution determines the performance of the circuit breaker. Despite the arc being the central and core issue of the interruption, it is far from being dominated, due to both the very high complexity of the underlying physics and the very sensitive dependence on a large number of small factors, which result in an extremely unpredictable and stochastic behavior.
1.1.2 Types of LVCB

According to a widely accepted conventional threshold, low voltage circuit breakers provide protection for circuit ratings of 1000 V or lower, whilst higher ratings fall under the scope of medium- and high-voltage breakers. The distinction is not only related to a conventional divide, for different and peculiar technological solutions are used to interrupt low, medium and high voltage arcs. Present day applications for LVCB are mainly found in residential electric distribution panels, industrial power supply centers and in main power supply panels, located in large buildings like offices, hospitals and shopping centers.

Low voltage circuit breakers are sometimes called *automatic* since the same product hosts the fault detection unit, the actuation unit, the interruption unit and its driving mechanism and the whole assembly acts as a unique, fully integrated device, with no external action or decision required in those cases when a fault current is detected. Actually, some of the above mentioned units coexist in the same product part. If both the energy required to detect a possible fault and the energy required to drive the opening mechanism, so to interrupt the current, are either stored inside the breaker itself, e.g., in charged springs, or found in the electric signal to be interrupted, e.g., by means of repulsive Lorentz forces induced by the electric current, then the breaker provides a passive protection and is termed *voltage independent*. Otherwise, and typically with reference to fault detection, if some external energy supply is required, e.g., to feed an electronic circuit carrying out some signal processing, then the breaker provides an active protection and is termed *voltage dependent*. Hybrid typologies also exist, with a limited amount of advanced, voltage dependent features, and with basic features still being voltage independent. The Standards prescribe the terms and conditions for employing voltage dependent or partially voltage dependent circuit breakers.

The family of low voltage circuit breakers contains many typologies which may heavily differ in size and ratings. Nonetheless, all kinds of LVCB share a large set of common features, so that their behavior may be studied with similar approaches. Most noticeably, all LVCB employ the same technology to break and exploit the same or similar technical concepts. Since the interruption phenomenon is highly nonlinear, the size factor inevitably induces problems of different relevance in different typologies.

To draw a precise and universally shared taxonomy of LVCB is not an easy task, for many different and not fully compatible criteria could be followed, according to either the functioning principle, or the type of protection to provide, or the categorization proposed by the Standards, or the traditional nomenclature diffused among the technical world, or even the commercial brands and internal product listings in use by the manufacturers. In the next sections we
will anyway try to provide a brief overview, for the sake of a better understanding of the object of our study.

A first distinction separates breakers for household and similar installations from those generally termed low voltage switchgear, or also simply low voltage circuit breakers, without a more precise definition (which may be a little misleading). The rationale in drawing the divide is that the breakers in the former category, differently from the others, must be such to be operated in a safe way by non expert and non trained users, and no special maintenance must be required. A conventional threshold also exists for the nominal current $I_n$, equal to 125 $A$ and bounding the former category from above and the latter from below. Actually, the high end segment of breakers for household and similar installations overlaps with the low end segment of “low voltage circuit breakers”.

1.1.3 Modular Circuit Breakers

The breakers in the category for household and similar installations are typically of very small and compact size, so to be arranged in large numbers inside the racks of distribution panels. A standard width is prescribed for a “module” and breakers (as well as their possible accessories) occupy half, one, or an integer number of modules. For this reason the breakers of this kind are also termed modular.

The historical tradition has lead to a further subdivision of modular circuit breakers, based on the type of protection to provide. A first sub-category includes those breakers only protecting from short-circuits and over-currents, as specified by the international Standard IEC 60898 [26] (or EN 60898 in the European Union). A widely diffused denomination is Miniature Circuit Breaker (MCB), in contrast with the others, bigger low voltage circuit breakers not for household. The size is actually the same as that of the other sub-category of modular circuit breakers, shortly hereafter described. The denomination is due to the large diffusion in electrical networks, which makes them the miniature circuit breakers par excellence.

The second sub-category includes those breakers only protecting from differential currents, as specified by the international Standard IEC 61008 [28] (or EN 61008 in the European Union). Many equivalent denominations are used for such breakers, including Residual Current Devices (RCD), Residual Current Breakers (RCB) and Residual Current Circuit Breakers (RCCB).

The third and final sub-category includes all-in-one devices, providing protection for short-circuits, over-currents and residual-currents, as specified by the
A modular circuit breaker (actually a MCB) is shown in Figure 1.1 (courtesy of ABB). The sample at hand is a single pole breaker, but the same concepts apply to multi-polar breakers. The breaker is connected to the rest of the circuit by two screwed terminals (1). Inside the breaker, the electric current follows a conducting path mainly made of copper. Along such a path, a bi-metal is first found (2). This device consists of two soldered laminae made of two metals with different thermal expansion coefficient. Joule heating induces a temperature rise field inside the bi-metal. Since the two laminae should elongate differently but are constrained to adhere, then they are forced to bend, so that the one with the highest thermal expansion coefficient takes the outer (and longer) part of the curved, deformed shape. The higher the electric current, the higher the bending. Over-currents produce a bending such that the bi-metal pulls (or pushes, in other configurations) a kinematic chain, triggering an opening maneuver. Owing to the heat capacity of materials, thermal phenomena are slow and inertial, so that the bi-metal technique is optimal for over-current detection but unsuited for short-circuit detection.

Proceeding along the conducting path, a solenoid is then found (3). The solenoid is part of an electro-mechanical actuator also consisting of a ferro-
magnetic plunger, located inside the solenoid and able to translate. Electric currents produce magnetic fields and the latter ones induce Lorentz forces on the plunger (due to its ferromagnetism), which is also counterbalanced by a spring. Short-circuits produce Lorentz forces strong enough to move the plunger against the spring and make it activate a kinematic chain, thus triggering an opening maneuver. Since electromagnetic phenomena are as quick as the speed of light, they are perfectly suited for short-circuit detection.

The final way to trigger the mechanism is by a deliberate user action, by means of a handle (4). Of course the handle is also used to close the circuit breaker. When closing the breaker, the user also provides the mechanical energy required to charge the main springs of the mechanism. Such energy will be unleashed during the opening maneuvers and will ultimately drive a mobile contact. Both the bi-metal and the solenoid only provide the activation energy of the mechanism, that is, the one necessary to remove a block that prevents the springs for discharging. Actually, the solenoid may also partially contribute to provide some energy to the moving part of the mechanism, at least during strong short-circuits.

Coming back to the conducting path, after the solenoid one finds the fixed contact (5) and the mobile contact (6), which in Figure 1.1 is shown in its open position. The opening of the contact produces a gap in the conducting path, which is temporarily filled by the electric arc (see §1.1.1) during the interruption. The mobile contact is electrically connected through a flexible wire to the final portion of the conducting path. Driven by Lorentz forces and pressure gradients, the arc is free to move beyond the contacts, along the arc runners (7), that is, two copper appendices, one electrically connected to the fixed and one to the mobile contact. The arc runners are separated by a diverging channel (8) whose purpose is to lengthen the arc and thus increase its resistance (see §1.1.6).
The arc ends its run in the \textit{de-ion chamber} (9), that is, a stack of metal plates having the purpose to break the arc into a series connection of smaller arcs, thus increasing its resistance (see §1.1.6). The denomination “de-ion chamber” is because the arc plasma is quenched inside it, so that ions and electrons recombine into neutral air §1.1.1). Another equivalent definition is \textit{extinguishing chamber}. Owing to their role and scope, the metal plates are also termed \textit{splitter plates}. They are made of ferromagnetic materials (typically iron) and present a V- or U-shaped notch in their portion toward the incoming arc. Both features are intended to locally modify the magnetic field, so to suck the arc inside the de-ion chamber. Some samples of stacks of splitter plates (courtesy of ABB) are shown in Figure 1.2.

The heat released from the arc induces strong density gradients in the surrounding fluid, increasing its pressure and making it expand. Pressure waves are produced, traveling around the arc chamber and finally finding their way out in the region (10), behind the de-ion chamber, and then through an \textit{exhaust hole} (11).

From the arc modeling standpoint and with some possible exception, modular circuit breakers are characterized by the arc motion from its ignition place, that is, the location of electric contacts, toward its extinction place, that is, the de-ion chamber. The possibility of a late entrance into, or that of an exit from the de-ion chamber would result in very severe and sometimes catastrophic effects on the interruption performance. The analysis of the arc dynamics naturally leads to an approach which couples fluid dynamics with electromagnetism, together with a series of non simple side issues, including radiation and the physical modeling of micro-scale phenomena with macro-scale effects (see §3).

\section*{1.1.4 Molded Case Circuit Breakers}

The whole category of “low voltage circuit breakers”, i.e., those for installations other than household or similar, is normalized by the international Standard IEC 60947 [27] (or EN 60947 in the European Union). Most of the breakers in this category provide protection from short-circuits and over-currents, even though residual current devices are not excluded. A further distinction may be drawn inside this large class, based on the size and performances of breakers.

A first sub-category includes the so-called \textit{Molded Case Circuit Breakers} (MCCB), see Figure 1.3, whose maximum interrupted current typically ranges from 100 \text{A} to 1600 \text{A}. This terminology stems from the fact that the breaker is fully contained into a plastic housing, which necessarily has to be removed to reach the internal parts. Figure 1.4 shows an internal view of a MCCB.
1.1. LOW VOLTAGE CIRCUIT BREAKERS

Figure 1.3: A molded case circuit breaker (courtesy of ABB).

(courtesy of ABB), where the main components can be easily recognized. A first important difference from the MCB is the vicinity of the arc ignition point, that is, the electrical contacts, with the arc extinction zone, that is, the splitter plate stack. This reason explains the absence of arc runners, which are not necessary since the mobile contact is opened inside or very close to the de-ion chamber.

The actuating mechanism is also completely different. In fact, in MCCB the mobile contact is allowed to rotate under the influence of the electromagnetic forces which grow up with the current in case of a short-circuit. It must be noted that Lorentz forces grow quadratically with current, so that, due to the higher current ratings, their influence in MCCB is far stronger than in the case of MCB. This feature is called “auto-protection”, since, above a certain current threshold, the contacts begin to open even without the consensus of the actuating mechanism, which acts on the mobile contacts through the shaft and is triggered by a magnetic or electronic release. Anyway, the auto-protection is not sufficient to ensure current interruption, since the contacts would tend to re-close when the opening (magnetic) force decreases as the current is extinguished. So, it is extremely important for the magnetic or electronic release to be fast enough to detect the fault and make the shaft rotate, in order to prevent the contacts from re-closing.

Apart from the opening mechanism, the extinguishing process of a MCCB is similar (if not identical) to that of a MCB. The arc fragmentation inside the arc
CHAPTER 1. INTRODUCTION

Figure 1.4: Inside view of a MCCB (courtesy of ABB).

Figure 1.5: MCCB with double interruption (courtesy of ABB).
chamber accomplishes two tasks. First, it guarantees a relatively high voltage across the breaker which, in turn, has a strong limiting effect (see §1.1.6) on the current which effectively flows inside the breaker. Second, it enhances the arc cooling preventing (or reducing the possibility of) a re-strike shortly after the current zero. It is not surprising, then, that the arc chambers are very similar in MCBs and MCCBs, up to an obvious scaling factor.

Frequently, a double interruption technique is adopted in MCCB (see Figure 1.5). Superior current limiting capabilities (see §1.1.6) and contact velocity are the main advantages of this configuration, in which two arcs in series connection enter inside the splitter plates more rapidly than an ordinary MCCB with single interruption. The arc voltage drop is doubled by the presence of two arcs instead of just one and, from the topological standpoint, the two mobile contacts substitute the flexible conductive connection, which is required in case of just one mobile contact and which has a somewhat unstable and difficultly predictable mechanical behavior, especially when damaged by the unavoidable wear. Lorentz forces acting upon each side of the support of the two contacts, together with the absence of the braking action of the flexible connection, explain the higher velocity in the contact opening observed with the double interruption. On the other hand, doubling the contacts means doubling the contact resistance when the breaker is closed to carry the current. This results in a higher Joule heating and in an increase of the temperature of the breaker, whose limits are strictly prescribed by the Standards. Therefore, as it often happens, a reasonable compromise has to be found.

1.1.5 Open Air Circuit Breakers

The high end segment of “low voltage circuit breakers” includes bigger devices than MCCB (see §1.1.4), which are termed (open) Air Circuit Breakers (ACB) and whose nominal current typically ranges from 2000 A to 6000 A. In contrast with MCCB, the housing of ACB does not enclose completely the breaker, whose arc chambers, for example, can be easily unplugged from the remaining structure, just unscrewing them. ACB differs from MCCB mainly in the actuating mechanism, which cannot be operated by hand through a lever. As a matter of fact, the energy required to operate the mechanism of ACB exceeds the average human capabilities. A charging lever is used to cumulate such an energy into the main springs of the mechanism. The breaker is operated by the user by means of suitable push buttons, which release the energy and trigger the mechanism.

Moreover, ACB are specifically designed to withstand (typically for 1 second) large short-circuit currents, in order to ensure the so-called selectivity,
i.e., allowing smaller and faster breakers, located immediately before the fault in the topological schematic of the electric network, to intervene first and selectively disconnect smaller portions of the electric network to be safeguarded. The ability of the breaker to keep its contacts closed is not an easy task, for the current can easily reach the value of $100\, kA$ and the corresponding repulsive, magnetic forces are considerably high. For this reason, many manufactures rely on magnetic simulations to optimize the shape of the conducting path, in order to reduce the opening forces as much as possible. One may notice that this goal is precisely the opposite of what is accomplished in a MCCB, which has to ignite (and hopefully) extinguish the arc as soon as possible.

Apart from the above mentioned differences, which are relevant to the size and the function of the breaker inside the electric network, there are no major differences in the concept and structure of an ACB and a MCCB. The same components, with the same functionality and different size, may be found inside the breakers of both the two sub-categories. To our knowledge, the previous

Figure 1.6: Air circuit breaker (courtesy of ABB).
distinctions between ACB and MCCB are not relevant with reference to the topics covered in this thesis.

1.1.6 Current Limitation

The most important feature of LVCB (at least those which are the subject of our study) is current limitation. An electrical device is termed a current limiter if it produces a lowering of the electric current with reference to its prospective value. This is usually accomplished by means of the insertion of an additional impedance into the electric network, so that the same voltage supply drives a smaller current. LVCB are limiting breakers and the additional impedance is ohmic and due to the arc itself, as later detailed.

The concept of current limitation is illustrated in Figure 1.7, where an oscillogram is shown relevant to a successful interruption by a T3 LVCB by ABB. The dashed, sinusoidal waveform is the prospective current, that is, the theoretical current in absence of any interruption. The actual, limited current is shown as a solid curve and is apparently much lower than its prospective

![Figure 1.7: Current limitation during a low voltage interruption.](image-url)
counterpart. The arc voltage wave form is also shown, and its steep increment, very close to the origin of the time axis, is the cause of the deflection of the limited from the prospective current. Current limitation has obviously a very beneficial effect when interrupting, since the current flowing through the breaker is lower, and thus less damaging, and since the current zero is forced to occur earlier. Thanks to current limitation, the ohmic power dissipated into the circuit breaker is considerably lower and it is possible to maintain the size of LVCB in the very small ranges currently found on the market.

The origin of current limitation has to be found in the high electrical resistance of the arc plasma. Differently from MV and HV, in LV the voltage drop through an arcing circuit breaker is of the same order of magnitude as the voltage supply. As a consequence, the breaker is a non negligible load during the interruption phase. Figure 1.8, from [61], shows a qualitative and approximate schematic of an arc column (upper part, A) and corresponding voltage drop (lower part, B). The resistance of the arc is due to two main contributions, namely an ohmic, distributed resistance and a concentrated amount, localized at arc root spots. The first contribution approximately follows Ohm’s second law, and it is thus proportional to the arc length and conductivity and inversely proportional to its cross sectional area. Owing to its ohmic nature, the relevant voltage drop is proportional to the current flowing, even though the proportionality is not necessarily linear. As a matter of fact, the higher the current the higher the Joule heating and thus the more conductive the arc.
Actually, the most effective contribution in LVCB is the second one, that is, the concentrated resistance of arc root spots. The very complex physics of arc roots will be briefly outlined in §3.9. The macroscopic effect is a cathodic voltage drop of approximately $10 - 15 \, V$ and a lower anodic voltage drop. Both contributions are found to be scarcely current dependent. Any cathode-anode pair is thus responsible for an approximately constant voltage drop of about $20 \, V$, which has to be compared with a voltage supply ranging some hundreds of volt. The de-ion chamber breaks the arc into a series connection of smaller parts, each one with its own cathode-anode pair and the relevant $20 \, V$ voltage drop. It is apparent that about ten splitter plates multiply the arc voltage drop (actually, its approximately current independent contribution) by a factor 10, making it definitely of the same order of magnitude as the voltage supply and causing current limitation.

From the above picture it follows the importance to bring the arc inside the de-ion chamber and forcing its permanence inside. This fact holds both in the case of modular circuit breakers, where typically the arc has to migrate into the de-ion chamber, and for MCCB and ACB, where typically the arc is ignited inside it. It also follows that the very same breaker may behave very differently in case the arc is kept inside the de-ion chamber or rather, for any physical cause, it manages to escape out or possibly it does not even enter inside the chamber.

1.2 State of the Art

The main underlying problem of current interruption in circuit breakers is the electric arc. There are many ways to approach such a phenomenon, and many levels of approximation within each approach.

The most physically oriented and, at least in theory, the most *a priori* approach is based on plasma physics. Obviously, many works exist in this field, including text books of various level of complexity and completeness, which are usually focused on plasma as an abstract subject, usually with the implicit reference to nuclear fusion or stellar plasma. Actually, the plasma behavior may change dramatically, depending on the physical conditions met in the application, and resulting in different possible regimes. Needless to say, the biggest theoretical difficulty, which is especially experienced at the technical level of industrial research, is to understand the kind of regime in the application at hand.

Despite the general, abstract picture of plasma is rich of references, a consistent and comprehensive treatment of the electric arc plasma, and particularly
in low voltage circuit breakers, is not available to the general public, at least to our experience. With this we mean that knowledge is possibly - actually very probably - available to many research centers held by industrial manufacturers and to cooperating academical institutions, but thus it is usually covered by the industrial secret and non disclosure agreements. Additionally, such level of theoretical knowledge is frequently not fully transmitted to the technical community of circuit breaker designers, despite belonging to the same industrial company and due to organizational overheads and different cultural backgrounds.

Many contributions to the description of the electric arc plasma originate from technical applications: plasma torches, high intensity discharge lamps, plasma cutting systems and plasma etching on semiconducting layers to fabricate integrated circuits, just to cite a few.

The world of circuit breakers is also providing contributions. Actually, very different kinds of circuit breakers exist in low, medium and high voltage, with consequently very different kinds of arcs found therein. The arc may take place in a gaseous atmosphere (SF$_6$, CO$_2$ or air) or in vacuum, involving a completely different physics. Arc current may assume very different values, from a few ampere, when a low voltage contactor is operated under normal, faultless conditions, to thousands of ampere, when an high-voltage generator circuit breaker is asked to clear a short-circuit. Accordingly, the role played by the current induced Lorentz force is completely different. Depending on the geometry of the breaker, the arc may be rather stationary until extinction or fast moving, involving a completely different fluid dynamic picture. Depending on different voltage ranges, dielectric rigidity may be, like in high-voltage, or not, like usually in low voltage, a main issue. Circuit breakers may be enclosed inside a bounded domain or be provided with exhausts to the external atmosphere, resulting, again, in a different flow picture. As a consequence, even though some methodological approaches and results may be ported from one kind of breaker to the other, a multitude of intrinsically different problems require different modeling and simulating solutions. Such a heterogeneous scenario is surely a practical obstacle to the industrial research on electric arcs, since different lines of research need being maintained and supported.

Historically, the first kind of circuit breakers to be modeled and simulated with reference to arcing phenomena belong to the high-voltage realm. This is easily understood, due to the high cost of both prototypes and testing of such very large breakers. On the other hand, low voltage circuit breakers are definitely cheaper products, and destructive testing is not necessarily an issue, from the economical standpoint. This partially explains why research on low voltage arcs is relatively younger and thus less developed than on high-voltage arcs. An additional factor is the 3D geometry of low voltage breakers, to be com-
pared with the axisymmetry of high-voltage ones, so that considerably larger numerical problems inescapably occur whenever a computational approach is attempted on the former.

In the last decade the scenario has changed, in favor of an increasingly pervasive diffusion of virtual prototyping. The need is acutely perceived by low voltage circuit breaker manufacturers to better understand the behavior of their products, so to shorten the time to market of new devices and start the experimental phase as late as possible, thus reducing the time required for the return of the relevant investment. Apart from purely management related reasons, merely trial-and-error aided engineering is often a blind approach, which leaves it difficult to extend the lessons learned to other developments. Also, a better comprehension of the hidden causes is necessary to define suitable remedial actions that frequently are required in order to increase the robustness of mass produced and relatively cheap products. Finally, the progressively increasing availability of computational resources, and with affordable costs for the industry, propelled the numerical simulation also in the case of low voltage.

The plasma theoretical description is naturally supplemented by a computational approach of multiphysical nature, meaning that a large set of partial differential equations must be solved, describing at least the flow, the electromagnetism, and the radiative heat transfer with a participating medium. Many authors have contributed to this line of research. Among them, we cite the classical works on low voltage switchgear by Lindmayer and his group [69, 77, 104] and by Wu et al. [168]. A multitude of conference papers are also available from other authors, witnessing the interest aroused by such an approach in the last decade. The universally adopted approach is magnetohydrodynamics, with weak coupling between Navier-Stokes and Maxwell equations. To our knowledge, the justification of the latter approach, which is related to the magnetic Reynolds number, is not explicitly addressed.

As regards the experimental approach, industrial level tests usually amount to sampling arc current and arc voltage signals during short-circuit tests. Measures are usually focused on the global scale, that is, the resolution is such to adequately measure current and voltage during the whole short-circuit. As a consequence, the current zero would require too fine measures for the standard current probes. In the framework of this thesis, tests will be refined so to include fine post-arc current measures and optical measures by means of fiber optics. Both the relevant techniques are nowadays very well assessed and do not represent a novelty as such. Particularly, the application of fiber optics to low voltage circuit breakers has already been introduced by McBride et al. [164, 93, 91, 92]. A benefit of our technique is the attempt to make the measurements as less intrusive as possible by drilling small holes into the breaker sidewalls, so not to modify dramatically the effect of gassing materials and the
gas outflow.

Other kinds of experimental methods have been employed to detect the position of the arc in a low voltage circuit breaker, but they will not be considered in this work. One alternative optical method is filming by means of a high-speed mechanical camera and a CCD camera (see, e.g., Brdys et al. [16]). The main drawback of this approach is the need to remove part of the breaker enclosure in order to make its interior visible to the camera. The perturbation induced is large, especially because the real influence of gassing materials is supposed to change consistently. Magnetic source localization by means of the solution to an inverse problem (i.e., from the effects - a measured magnetic field - to find the causes, like an unknown current distribution) is a well known and aged technique (see, e.g., Model and Trahms [98]), and it has been applied to low voltage breakers as well, by Mercier et al. [95, 96], Cajal et al. [19], Debel-lunt et al. [32] and Brdys et al. [17, 16], with an increasing level of refinement and complexity.

The sensing device is constituted by an array of Hall effect probes and the sought-for electric arc shape is resumed to a polyline or a more elaborated bouquet of segments, which is still too oversimplified with reference to the outcome of fiber optics. Furthermore, ferromagnetic splitter plates add a non negligible contribution to the magnetic field, superimposed to that directly induced by the arc. As a consequence, to correctly identify the arc position in a real, industrial case becomes a complex problem. The empirical level of knowledge of many important aspects, like the precise definition of the B-H curve of ferromagnetic materials used in LVCB, is also an issue. A reliable solution requires a sophisticated finite element solver, and the problem may become computationally intensive.

The last approach dealt with in this thesis is black box modeling. Particularly, we are interested in local models for the current zero period. A large review on this subject is reported in §5.5, based on the works by CIGRE. The first works appeared in the 30’s and in the 40’s and most of the late contributions on this field are rather old, dating back to the 70’s and the 80’s. Only a few works appeared at the beginning of the 21st century. Historically, high-voltage has been the addressed range in the case of black box modeling. Purely electrical phenomena are concerned in local black box arc models, with all the other physics being hidden inside some suitable parameters to be identified, based on an experimental test.

Also global models have been produced, meaning that the overall arc history is addressed, starting from contact opening and arc ignition up to the current zero and possibly over, in case of a re-ignition. The mechanical motion of contacts, possibly driven by Lorentz forces, thermal phenomena and also other
physical issues may be present in global black box arc models, supplementing purely electrical phenomena. Non electrical phenomena appear in the form of lumped parameter terms allowing for analytical solution or deduced from the best fit of experimental data. All of this models are rather old, and may be conceived as a sort of primitive and miniature multiphysical approach, suited for the very limited computational power of past computers. A possible use of such models is to describe an arcing circuit breaker inside an electrical network, for simulations intended to cover a long time period and with the focus on the rest of the network rather than on the circuit breaker itself. This goal may be alluring in high-voltage networks, where minutes may pass before a detected fault is cleared by the circuit breaker, but it is less interesting in low voltage networks, where a fault is cleared practically immediately after its detection. Such models will not be considered in this thesis.

1.3 Scope of Work

The current knowledge on electric arcs in current interruption is well below the level required to dominate the problem. This thesis is intended to provide a contribution in the modeling and possibly in the simulation of such a complex phenomenon. The object of our study are real, low voltage circuit breakers, and the final goal is to deliver results which are of interest for the technical community of industrial manufacturers.

From the experimental standpoint, a refinement of currently available measuring techniques is concerned in this study. Particularly, we propose to focus on the current zero region, which we want to investigate by means of refined current measures. Moreover, the adoption of a slightly intrusive optical method, such as fiber optics, is intended to overcome the drawbacks of fast camera filming. We also intend to cross-match electrical and optical measurements, in order to gain a deeper insight in the arc dynamics and to develop a key to interpret test lab oscillograms. Suitable techniques for handling fine measures will be addressed and tested.

Based on the positive experience in high-voltage, we are interested in defining some performance evaluators, correlated with the successfulness in current interruption and to hold in low voltage. The main overall idea is to have a graded response, richer and finer than the simple success or failure verdict, which is currently the case in industrial testing.

Black box modeling is a potentially very attractive line of research, due to its extremely limited computational burden. This study is aimed to verify whether or not such an approach may be profitably adopted in low voltage
circuit breakers and to what an extent. Particularly, we would like to assess
whether black box models can be descriptive, that is, if they can adhere to ex-
perimental data, or also be even predictive, that is, if they can reliably simulate
different conditions than those used for the identification of their parameters.
In the end, the base question is about the extremely simplified nature of black
box models, which are such to resume very complex physical issues inside nu-
merical parameters, to be identified with an experimental test. Since no model
has been proposed in low voltage, we would like to understand whether there
is some benefit to be gained or rather whether this is the outcome of an un-
reported story of failures, maybe due to a deficiency of the approach when
applied to the peculiar conditions of low voltage.

In order to work with black box models, a suitable parameter identification
method is required and will become part of our study. The method should be
general enough to be applicable to the whole family of black box arc models to
be considered, so to allow for comparisons, and fast enough to allow processing a
large amount of experimental data. The method should also be such to account
for bounds on parameters, which can be posed by physical considerations or
simply for the sake of convenience, in order to prevent an excessive variability
of parameters.

This thesis intends to review the theory of electric arcs. The problem is
vast and complex, mainly concerned with plasma physics but also ranging
over other special problems, such as field-enhanced thermionic emission. A
first reason is to find a justification of the observed empirical behavior, as
well as a physically consistent modeling with which to attempt a predictive
approach. It is also believed that a better level of physical comprehension
could be helpful in designing better circuit breakers, maybe because it can
contribute in understanding the inner reasons for failures or successes.

Furthermore, it is important to recognize which physical phenomena fall
under the resolution power of a given physical level of description, so to under-
stand what to expect from a computational approach based on any given set of
physical equations. Particularly, we want to delineate the physical bases and
the mathematical formulation of magnetohydrodynamics, which is currently
the most attractive approach for arc simulation. Since most of the implemen-
tations are based on weak coupling of Navier-Stokes and Maxwell equations,
we are interested in checking whether this approach is reasonable and justified
for low voltage circuit breakers.

Based on the experience in high-voltage, the role of non equilibrium theory
is expected to play an important role in electric arcs, especially in conditions
close to extinction. Our study is aimed at checking if this is also the case in
low voltage, and to trace back non equilibrium dominated phenomena to their
1.4 Outline of the Thesis

In this introduction the object of our study is presented. A brief description of low voltage circuit breakers is given, together with the phenomenon of the electric arc and its central role in current interruption. Moreover, the interest and scope of our research is outlined.

In Chapter 2, the experimental part of our work is described, mainly consisting of electrical and optical measures, their post-processing techniques and the analysis of results, with the definition of performance evaluators and their statistical spread.

In Chapter 3, the theoretical base of the electric arc plasma is presented. The physics is first introduced at the particle level, then turned into a kinetic theory, and finally into a fluid theory, ultimately leading to magnetohydrodynamics. Radiation, material ablation and arc root modeling are treated. Local thermal equilibrium or, contrarily, non equilibrium conditions are defined and assessed in the case of low voltage circuit breakers.

In Chapter 4, the mathematical structure of the magnetohydrodynamic problem is first outlined in its strong form and then cast into weak form and numerically solved by means the Galerkin method. Particularly, the finite volume method is adopted for fluid dynamics and radiative transfer, the classical, node-based finite element method is used for electrostatics and finally the edge finite element method for magnetostatics. Some preliminary results are presented, showing the potentialities of the computational approach.

In Chapter 5, the black box approach is presented. First the arc-network interaction problem is mathematically set and solved. Then existing black box models are reviewed and a model for low voltage circuit breakers is proposed. A general model identification routine is proposed. A comparison of our model with the experimental data is shown.

In Chapter 6, conclusions are drawn concerning the work done and the results presented in this thesis. Recommendations for future research are also provided.
Chapter 2

Experimental Behavior

2.1 Summary

The experimental behavior of low voltage arcs is dealt with in this chapter, providing the empirical reference for the subsequent modeling and simulation. All the measurements have been carried out in ABB S.p.A. - Automation Products Division testing laboratories and by means of either standard testing instrumentation or specially made testing devices.

First, electrical measurements are introduced, being current and voltage measurements versus time, during short-circuit tests. A particularly accurate current sensor is used to measure post arc currents, which are in the range of a few ampere. Such a sensor is not standard in industrial testing laboratories and has the special feature to virtually immediately recover after high currents (well above its saturation threshold) and yields an accurate current measure in its very low range. The testing network and the testing conditions are described and the conditions of a lumped parameter modeling are positively assessed. Suitable experiments have been carried out in order to correctly model the network as a RLC system, and in particular to identify the value of the stray capacitances. The latter are necessary in order to reproduce the transient recovery voltage oscillations which are observed soon after the current zero, when the breaker voltage drop switches from the arc voltage to the supply voltage.

Electrical measurements need to be suitably post-processed in order to be
usefully handled. Apart from the consistent merging of measurements coming from different sources and the removal of possible offsets, the core issue is signal regularization. For instance, other electrical quantities deduced from arc current and arc voltage, such as arc conductance and its time derivative, as well as the time derivatives of arc current and arc voltage, require regular signals. Noise removal is accomplished by means of a procedure starting with a Fourier transform based filter and being improved by Savitsky-Golay filter, which basically consists of a local best fit polynomial approximation and which is shown to be a moving average linear filter. The superior benefit of Savitsky-Golay filtering is the immediate availability of accurate estimates of the time derivatives of the filtered signals, by means of the derivatives of the local approximating polynomial.

Optical measures are also possible by means of fiber optics. A specifically developed experimental setup, along with a suitable post-processing software, has been used in order to film the motion of the arc in low voltage circuit breakers under short-circuit tests and having an array of holes drilled in their sidewalls, where optical fibers are hosted. The technique allows studying, as a special case by-product, the arc root dynamics. Particularly, experimental evidence is found of a stationary time before the arc foot begins to move. This fact suggests that a thermally inertial phenomenon governs the electron emission from the electrode into the arc plasma. On the other hand, a field-enhanced thermionic emission (Schottky effect) in low voltage circuit breakers is suggested by the basic consideration that the electrode melting temperature would be too low to allow a purely thermionic emission (Richardson’s law) of the observed current rates. The required, strong electric fields at arc roots could be justified by charge density layers close to the electrodes and of very thin depth, as predicted by Debye shielding theory.

Finally, the issue of evaluating the circuit breaker performance is addressed by the definition of a series of numerical evaluators, whose value is intended to be correlated with the quality of the current interruption under short-circuit tests. No correlation is found when the global, or macroscopic, behavior of the arc is concerned, i.e., when the evaluators are based on data collected over a long period of time relatively to the test duration. On the other hand, good local, or microscopic, evaluators are found, all of them based on data collected at the current zero or in its immediate surroundings. A remarkably sharp distinction between successful interruptions and failures is observed below or above, respectively, a given threshold value. The reason why the current slope before the current zero is not a good performance evaluator in low voltage (which is the case in high-voltage) is theoretically explained. This part can also be found in journal paper [9]. The stability and repeatability of short-circuit tests under nominally identical conditions, both for the breaker and for
the testing network, is also investigated. A very large scattering is observed in microscopic performance evaluators, even though a success/failure divide can still be defined. This fact is rather discouraging, though not particularly surprising, and simply witnesses the well-known, remarkable unpredictability and highly stochastic behavior of the electric arc.

2.2 Electrical Measurements

The numerical simulation of short-circuit phenomena has greatly improved over the last years, especially thanks to the computational power gained with present-day computers. Nonetheless, we are considerably far away from developing an interrupting device relying only on theoretical computations, and testing in power laboratories still has a major role while developing a new product. This is the main reason that leads manufacturers to provide themselves, when economically feasible, with a power laboratory to carry out their own measurements. ABB S.p.A. - Automation Products Division owns a testing network equipped with three generators, including one which is capable of supplying a short-circuit power of 2800 MVA. The generated power can supply various types of loads, such as transformers, induction motors, capacitors, resistors and reactors, which makes it possible to obtain the widest range of conditions encountered in electrical installations.

We have carried out all measurements over T3 low voltage circuit breakers, from ABB S.p.A. - Automation Products Division. The breaker belongs to the category of MCCB (see §1.1.4). The International Standard which regulates the testing conditions for LVCB is IEC 60947-2 [27], or EN 60947-2 for the European market. In the American market the UL 489 [66] Standard is used. Beside these basic requirements, ABB power lab fulfills the requirements for what concerns the electric measurements and the corresponding electronic devices. Among the great variety of signals that can be registered during a short-circuit, the current which flows in each phase of the breaker under test and the phase voltage play a key role. Other quantities, such as the arc conductance, energy and power can be readily deduced off-line with elementary operations. The accuracy of the measured signals depends on what we want to investigate. Thus, if we are interested in, say, the arc energy over the entire interruption process, there is no need to use a probe to detect a current of the order of ampere. On the other hand, if we want to calculate the arc conductance over a interval of ±200 µs around the current zero, special techniques have to be taken into account. As a matter of fact, in this thesis we will focus mainly on this kind of problems.

In the next sections we give a brief overview of the ABB testing network
CHAPTER 2. EXPERIMENTAL BEHAVIOR

for low voltage tests, including the measurements that have been carried out to determine the transient behavior during current interruption. The main purpose of this analysis is to produce a lumped-parameters model to be eventually combined with an appropriate arc model to gather precious information about the arc-network interaction. Most of the data was already available from the technical handbooks of the electrical equipment of the laboratory. Other information, such as the value of the stray capacitances, have been deduced from new experiments carried out by ourselves and ABB laboratory personnel.

2.2.1 The Lumped Parameter Approximation

The electromagnetic phenomena taking place in the testing network are rigorously described by Maxwell equations (see §3.4.3), that is, field equations. Nonetheless, the simpler, lumped parameter approximation could be adequate for describing the circuit behavior. It is well known (see, e.g., Magid [83]), that the lumped parameter approximation is equivalent to a quasi-static approximation of Maxwell equations. Precisely, all time varying electromagnetic quantities are expressed in Maclaurin series expansions with reference to their frequency, and then the expansions are truncated to the first order. It follows that only low frequency phenomena can be caught and correctly modeled. Particularly, the lumped parameter approximation is acceptable as long as the shortest electromagnetic wavelength of interest (i.e., the fastest harmonic of interest) is longer than the typical length of the circuit at hand. In this case the structure is said to be electromagnetically small [25]. In the case at hand, in order to remove the experimental noise from the measured signals, we have adopted a 100kHz cut-off frequency $B$ (see §2.3.2), corresponding to a wavelength $\lambda = c/B \approx 3\text{ km}$. Conventionally, an electromagnetically small structure is such if

$$l < \frac{1}{10} \lambda,$$

where $l$ is its characteristic length [25]. The size of ABB test lab is of the order of some tenths of meters, and thus it is shorter than the conventional $\lambda/10 = 300\text{ m}$ threshold.

In conclusion, the lumped parameter approximation introduces no significant approximation in addition to the noise removal filtering adopted. For this reason, in the rest of the thesis the testing network will always be described as a lumped parameter circuit, even with reference to the multiphysical description of the arc as a distributed parameter phenomenon.
2.2.2 Overview of the Test Circuit

Figure 2.1 shows the basic schematic representation of the test network we used for our experiments, and Table 2.1 reports a summary of the network parameters to be later used in the simulations. A three-phase generator (1), driven by a DC induction motor, supplies the primary side of a medium-to-low voltage transformer (4) which, in turns, provides the power that is necessary to perform the short-circuit tests. Several elements are inserted along the circuit from the generator (1) to the devices under test (9). Basically, we can divide them into three categories: first, protection elements such as medium-voltage circuit breakers (2); second, measuring instruments such as shunts for current (6) and dividers for voltage (8); third, calibration systems like adjustable reactors (3) and resistors (7) to obtain the desired prospective current, phase-to-phase voltage and power factor. The signals coming from shunts and voltage dividers are transmitted with fiber optics to an acquisition system with a sampling rate up to 10 MHz.

Figure 2.1: Schematic representation of the test network: 1 - three-phase generator, 2 - back-up circuit breaker, 3 - air core reactors, 4 - three-phase transformer, 5 - short-circuit making switch, 6 - non inductive shunts for current measurement, 7 - resistors, 8 - dividers for voltage measurement, 9 - test object, 10 - post-arc current sensor.

Several measurements have been done in order to calculate the total resistance and reactance of the network. Some data can be found in the technical handbooks of the various devices, but other information, such as the transient response to the current interruption, requires ad hoc experiments. A summary is reported hereafter.

The power plant has two different shunts for current measurements, and their internal resistances have to be taken into account for the network pa-
**Generator**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_n$ Nominal voltage</td>
<td>4900 V</td>
</tr>
<tr>
<td>$S_n$ Apparent power</td>
<td>32.5 MVA</td>
</tr>
<tr>
<td>$R$ Internal resistance</td>
<td>0.0033 Ω</td>
</tr>
<tr>
<td>$X''$ Subtransient reactance</td>
<td>0.233 Ω</td>
</tr>
<tr>
<td>$X'$ Transient reactance</td>
<td>0.357 Ω</td>
</tr>
<tr>
<td>$X$ Synchronous reactance</td>
<td>4.1 Ω</td>
</tr>
</tbody>
</table>

**Transformer**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{eq}$ Equivalent resistance</td>
<td>35.2 mΩ</td>
</tr>
<tr>
<td>$X_{eq}$ Equivalent reactance</td>
<td>0.141 Ω</td>
</tr>
<tr>
<td>$R_M$ Magnetizing resistance</td>
<td>748.7 mΩ</td>
</tr>
<tr>
<td>$X_M$ Magnetizing reactance</td>
<td>249.6 Ω</td>
</tr>
</tbody>
</table>

**Connection bars - Generator → Transformer**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$ Total resistance</td>
<td>1.02 mΩ</td>
</tr>
<tr>
<td>$X$ Total reactance</td>
<td>6.25 Ω</td>
</tr>
</tbody>
</table>

**Connection bars - Transformer → Breaker**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$ Total resistance</td>
<td>0.2 mΩ</td>
</tr>
<tr>
<td>$X$ Total reactance</td>
<td>0.7 Ω</td>
</tr>
</tbody>
</table>

Table 2.1: Summary of the network parameters

Parameter calculation. From the technical sheets we gather that the shunts for currents up to 100 kA have a resistance of 0.02 mΩ, while the shunts for currents up to 20 kA have a resistance of 0.31 mΩ.

Shunts and dividers measure the arc current and the arc voltage during the whole arcing period. The current ranges from zero to several kiloampéres,
and for this reason, the resolution of the signal is not high enough to measure post-arc currents, which typically range from zero up to ten or twenty ampere. To overcome this problem, a measuring device which is able to measure small currents with a high resolution and capable of withstanding large peak currents has been developed (see Figure 2.2). In contrast to commonly used current monitors, saturation of the core of the transformer at high pulses of current is prevented by shunting the turns of the transformer by anti-parallel fast-recovery diodes. The device has a sensitivity of $1 \text{mV/A}$, is linear within 1% in the range of $\pm 200 \text{A}$, and is designed for sinusoidal fault currents up to 150 $\text{kA}$. The upper cut-off frequency is 234 kHz.

### 2.2.3 The generator

The Generator (see Figure 2.3) is a two pole synchronous machine driven by an induction motor, which makes the generator spin asynchronously. For the parameter calculation, we assume a constant frequency of 50 Hz even if, as shown later, the real value is a bit lower. The internal resistance of the generator and its inductance can be found from the technical guide of the generator. Their values are, respectively, 3.3 $\text{m} \Omega$ and 0.33 $\text{mH}$ per phase. When dealing with three-phase balanced circuits, a *single-line* representation can be used (see Figure 2.4). This simplification, which consists in drawing only one phase of the

![Figure 2.3: Overview of the generator.](image-url)
network, is allowed because in three-phase balanced circuits all the currents, voltages and circuit elements are symmetrical. Thus, showing only one phase, it is possible to represent the three-phase system. The main characteristics of the generator can be found in its technical handbook. Other quantities are readily calculated, such as the real power, \( P_n = (S_n \cdot \cos \varphi)/3 \), and the reactive power, \( Q_n = (S_n \cdot \sin \varphi)/3 \), being \( S_n \) the apparent power (see Table 2.1). The per-unit values for the resistance and the inductances are calculated with reference to \( R_{\text{base}} = V_n^2/P_n \) and \( X_{\text{base}} = V_n^2/Q_n \).

### 2.2.4 The transformer

The nominal power of the transformer is 200 MVA; the parameters \( X_{\text{eq}}, R_{\text{eq}}, X_m \) and \( R_m \) of the transformer (see fig. 2.5) were calculated with short-circuit
and open-circuit tests. These experiments were carried out in previous years and the results have been used to derive the parameters of the equivalent circuit. The results are referred to the primary side of the transformer. During short-circuit tests, the transformer was connected in Y-Y configuration, with a transformation ratio of 8500/554. Denoting with \( a \), \( b \), and \( c \) the three phases, we have \( I_a = 411 \, \text{A} \), \( I_b = 440 \, \text{A} \), and \( I_c = 411 \, \text{A} \) for the currents and \( V_a = 62 \, \text{V} \), \( V_b = 60.5 \, \text{V} \) and \( V_c = 61.2 \, \text{V} \) for the voltages.

From these results we find that the mean current (over the three phases) and the mean voltage are 420 A and 61.2 V, respectively. The mean power was measured during the test, and the value of 6.2 kW was found. With these results
we can easily calculate the equivalent resistance \( R_{eq} = P/I_m^2 \), the equivalent impedance \( Z_{eq} = V_m/I_m \) and the equivalent reactance \( X_{eq} = (Z_{eq}^2 - R_{eq}^2)^{1/2} \).

From the open circuit tests, also carried out in previous years, the magnetizing inductance \( X_M \) and the magnetizing resistance \( R_M \) of the transformer can be calculated. During these tests, the transformer was again connected in Y-Y configuration with a transformation ratio of 8500/554. The measured voltage, power and current during the open-circuit test were, respectively, 8500 V, 96.5 kW and 31 A. The magnetizing inductance and resistance are easy evaluated as \( R_M = V^2/P \) and

\[
X_M = \frac{1}{\sqrt{(I/V)^2 - 1/R_M}}.
\]

### 2.2.5 Connection bars

In the power plant there are two main bus bar systems. The first one carries the current from the generator to the primary side of the transformer; the second runs from the secondary side of the transformer to the adjustable-resistors bank. The final connection with the breaker under test is provided by cables whose diameter vary according to the prospective current and it is explicitly prescribed by International Standards such as IEC 60947.

### 2.2.6 Air-core reactors and resistors

In order to get the desired prospective current and power factor during the various tests, the value of the air-core reactors (connected to the primary side of the transformer; see Figure 2.7) and the value of the resistors (connected to secondary side; see Figure 2.8) vary upon a wide set of values. For a typical UL489 test at 8.7 kA, 480 V, \( \cos \varphi = 0.5 \), we have the nominal value of 14.5 m\( \Omega \) for the resistance and 1490 m\( \Omega \) for the reactance. All of the tests carried out in the framework of this thesis are relevant to the aforementioned values, the only possibly differing datum being the voltage supply.

### 2.2.7 Total resistance and inductance at secondary side

Given the transformation ratio of 8500/554 we can calculate the total impedance as seen from the secondary side for a test at 8.7 kA, 480 V and \( \cos \varphi = 0.5 \) by summing up all the equivalent impedances at the primary side, transforming them to the secondary side and adding the impedances from the transformer to the breaker. We then find a total resistance of 15.2 m\( \Omega \) and a total reactance
2.2. ELECTRICAL MEASUREMENTS

Figure 2.7: Overview of the adjustable reactors inserted just before the device under test.
Figure 2.8: Overview of the adjustable resistors, inserted between the secondary side of the transformer and the device under test.
of 21.9 mΩ. With the values of the total resistance and inductance at the secondary side of the transformer, the equivalent model shown in Figure 2.9 can be used to simplify the circuit of the network and to approximate the behavior of the distributed system.

Figure 2.9: Equivalent schematic representation of the network. The values of resistances and reactances stem from calculations and refer to a 8.7 kA, 480 V, cos ϕ = 0.5 test.

### 2.2.8 Network Parameter Calculation

In order to verify the values found for the equivalent resistance and reactance of each phase, a measurement of a single-phase current transient has been carried out. We used an ABB MV vacuum circuit breaker, whose contacts were kept closed until the transient had vanished, i.e. after about 60 ms. At this instant, the current is interrupted by the breaker. Figure 2.10 shows this kind of measurement, where the solid line is the current, while the dashed line is the voltage across the breaker, which is zero (or close to) until the current is interrupted. The transient recovery voltage (TRV) contains valuable information about the stray capacitance of the network and will be analyzed later.

The network equation which corresponds to our simplified case of Figure 2.9 reads

\[ L \frac{di}{dt} + Ri = V_s \sin(\omega t + \theta), \]

(2.1)

where \( i(t) \) and \( V_s \) are the current signal and the peak value of the supply voltage, respectively; \( \theta \) determines the value of the supply voltage at the beginning of the short-circuit. Solving (2.1) with \( i(t = 0) = 0 \) as initial condition, we have

\[ i(t) = \frac{V_s}{R} \cos \varphi \left[ \sin(\omega t + \theta) - \sin(\theta - \varphi) e^{-\omega t/\tan \varphi} \right], \]
where $\varphi = \arctan \omega L/R$. This expression has four unknown parameters, namely $V_s$, $R$, $\cos \varphi$ and $\theta$, which can be easily fitted from an experimental curve. Results can be found in Table 2.2.

In particular, we are interested in a comparison between the experimental (i.e., fitted from an experiment) impedance and the corresponding value calculated summing up all the contributions along the conduction line from

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos \varphi$</td>
<td>0.54</td>
</tr>
<tr>
<td>Total resistance</td>
<td>15.72 m$\Omega$</td>
</tr>
<tr>
<td>Total reactance</td>
<td>24.50 m$\Omega$</td>
</tr>
<tr>
<td>Closing angle</td>
<td>0.104 rad</td>
</tr>
<tr>
<td>Voltage supply</td>
<td>480 V</td>
</tr>
<tr>
<td>Prospective current</td>
<td>8.26 kA</td>
</tr>
<tr>
<td>Frequency</td>
<td>49.65 Hz</td>
</tr>
</tbody>
</table>

Table 2.2: Results of the identification for the network parameters, based on the current transient.
the generator to the breaker. From Table 2.3, we see that the experimental resistance and reactance overestimate the corresponding theoretical values. Anyway, the air-core reactors have a non-zero resistance, which has not been considered in the calculations, and the resistance at the secondary side, due to the actual shape of the connections, introduce some inductance. Therefore we conclude that the calculations are in good agreement with the measurements.

The model of the network shown in Figure 2.9 is satisfactory as far as only large-scale current transients are considered. As a matter of fact, when investigating voltage transients one needs to take into account the stray capacitances that are always present in every real network. The same experiment used to

### Table 2.3: Comparison between calculated and experimental (i.e. fitted from measures) total resistance and reactance of the network.

<table>
<thead>
<tr>
<th></th>
<th>Calc.</th>
<th>Exp.</th>
<th>Comp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R [m\Omega]$</td>
<td>Total resistance</td>
<td>15.2</td>
<td>15.72</td>
</tr>
<tr>
<td>$X [m\Omega]$</td>
<td>Total reactance</td>
<td>21.9</td>
<td>24.50</td>
</tr>
</tbody>
</table>

Figure 2.11: Voltage transient.
analyze the current transient can now be used to see the characteristic voltage transient across the breaker after current interruption; see Figure 2.11. The oscillations of the voltage strongly depend on how rapidly the current is interrupted. Later in this chapter we will show that after the current zero a small current is still flowing into the breaker (post-arc current) and that the quality of the interruption is greater when a small post-arc current is observed.

In order to make the voltage transient be nearly independent of the interruption process, the post-arc current has to be reduced as much as possible. This is the reason that has led us to use a vacuum circuit breaker which, being rated for Medium-Voltage (MV) interruptions, makes the current extinguish almost instantly. In order to take into account this behavior, the equivalent circuit of the network can be modified as shown in Figure 2.12. An ideal interruption process, i.e., with instantaneous opening of the breaker and without post-arc current, has been simulated with MatLab-Simulink. Comparing the simulated TRV with the experimental signal (see Figure 2.13) we find that the stray capacitance and resistance are, respectively, 0.487 $\mu F$ and 40 $\Omega$. The frequency of the oscillations is 17.8 $kHz$. From Figure 2.13 we see that the absolute value of the experimental TRV is lower than the simulated one. This phenomenon, very difficult to be modeled in a simple way, is called depression and is caused by the rearrangement of the electromagnetic field inside the generator after the short-circuit current has been interrupted. This result represents a serious obstacle in our original idea of exploring the interruption performances of a breaker by simply coupling a lumped-parameter model of the network with a suitable model of the arc. The main reason is due to the crucial role that the current zero phenomena play in current interruption and thermal re-ignition (see §5). For example, an incorrect modeling of the transient recovery voltage from the network point of view would lead to an unacceptable error in the

![Figure 2.12: Schematic of the equivalent network including the stray capacitance and resistance.](image)
calculation of the energy put into the extinguishing arc immediately after the current zero. Thus, any forecast about the outcome of the interruption would be meaningless even with a realistic arc model.

### 2.2.9 Current Zero Behavior

The typical shape of the time history of arc current in a neighborhood of one of its roots is shown in Figure 2.14. Many oscillograms and current zeros have been analyzed, all relevant to the same testing network (although for different testing conditions), all exhibiting very similar features. Arc current manifestly approaches the zero with a linear descent (in magnitude) as long as current is higher than a threshold whose order of magnitude is of some ampere. Below such a threshold, the concavity of the current curve is no longer null, the slope progressively and non negligibly decreases and the final descent to the zero is less than linear. It must be noted that such a behavior is not necessarily common to different circuits, conditions and switching devices. For instance, the exhibited behavior before current zero is opposite the so-called “current chopping”, typical for medium-voltage vacuum circuit breakers interrupting small inductive currents [113] and characterized by a more than linear final descent to the zero (usually accompanied by high frequency oscillations).

### 2.3 Post-processing Techniques

Experimental current and voltage time histories need to be suitably post-processed in order to yield useful information for accurate analyses. Particularly, the final goal to accurately describe a bilateral neighborhood of the current zero, and with differentiable signals, must be accounted for. A set of
actions have been defined accordingly, in the form of a post-processing software routine to be executed off-line and automatically (that is, not requiring *ad hoc* user input).

### 2.3.1 Offset removal and signal merging

Any possible offset must be removed from current signals. This is particularly important in the case of the signal from the high sensitivity current probe, since post-arc currents will be seen to be crucial in the interrupting process. Under the assumption that the probe is such not to modify the zero of its scale during the test duration, the average value of a suitably extracted current signal portion, where current would be expected to be rigorously null, can be used to detect a possible offset and eventually correct (by subtraction) the relevant signal. To this purpose, a good time interval is seen to be the one before triggering the current probe.

The output signals from the shunts and from the post-arc current measur-
ing device have to be seamlessly merged into a single signal plot. The most straightforward implementation implies choosing a current threshold and keeping higher currents from the shunts and lower currents from the high sensitivity current probe. Subsequently applied smoothing filters will provide a suitably regular junction in between the two signals, now merged together.

The current and voltage signals so obtained are still noisy, which would considerably spoil other quantities, such as conductance, to be deduced from them and which absolutely prevents them from being differentiated. Noise removal can be accomplished in several ways.

\subsection{2.3.2 Filtering}

In our implementation, first a low pass Gauss filter is applied to damp out high frequency spikes. We denote by \( \{(t_k, x_k)\}_{k \in K} \), the input sampled time varying signal, where \( t \) is time and \( K \) is a suitable index set. Then the discrete FFT spectrum \( \{(f_j, X_j)\}_{j \in J} \), is computed, where \( f \) is the frequency and \( J \) is a suitable index set relevant to \( K \). A new frequency domain spectrum \( \{(f_j, Y_j)\}_{j \in J} \), with \( Y_j = X_j \cdot \exp(-f_j^2/B^2) \), is produced and then brought back to the time domain by means of the inverse discrete FFT. The output sampled temporal signal \( \{(t_k, y_k)\}_{k \in K} \) replaces the original signal, yielding a noise reduction which is dependent on the cut-off frequency \( B \), set to 100 kHz.

After the application of the Fourier transform based Gauss filter, noise removal is further improved by Savitsky-Golay filtering \[129\], which possesses the nice property to yield regular time derivatives of the processed signal. The basic idea is to locally approximate data by the least squares fitting polynomial of a specified fixed degree \( d \) and computed with reference to a moving window of specified fixed width. Such a window is chosen so to contain an odd number of samples \( m \), that is, the central sample itself and two equally long tails, one before and one after. The fitting polynomial is then used to define the local value of the smoothed signal and of its derivatives.

We denote again by \( \{(t_k, x_k)\}_{k \in K} \) the input signal, i.e., the output from the Gauss filter, and we assume samples to be equally spaced by a time step \( h \). The latter hypothesis of uniform sampling could also be dropped, so to cover more general cases (though not particularly frequent in practice), but only sacrificing the considerable speed up hereafter described. The following shows how to efficiently deduce the smoothed output signal \( \{(t_k, y_k)\}_{k \in K} \) and its first derivative \( \{(t_k, \dot{y}_k)\}_{k \in K} \). We set \( l := (m - 1)/2 \) and focus on the general \( k \)-th sample, assuming to be far away from the edges of the time history, so that a complete window \( W_k := \{(t_n, x_n)\}_{n=k-l}^{k+l} \) centered on \( t_k \) can always be defined. This is not a major concern, because time histories are usually longer...
than necessary and may be clipped from their extremities.

A family of affinities \( \{A_k\} \in K \) is defined, where the general application

\[
A_k : \mathbb{R} \rightarrow \mathbb{Z} \subset \mathbb{R}
\]

is such that

\[
t \mapsto A_k(t) = (t - t_k)/h = z(t).
\]

Each affinity acts as a local, linear rescale of time, so that

\[
A(W_k) = \{-l, \ldots, -1, 0, 1, \ldots, +l\} =: S, \quad \forall k \in K,
\]

that means, the time window is always mapped, regardless the value of \( k \), to the same subset \( S \) of the ring of integers \( \mathbb{Z} \) (even though a same time instant \( t_n \) will be mapped to different elements of \( S \), depending on the relative position inside the moving window \( W_k \), for different values of \( k \)).

The least squares fitting polynomial \( p_k(z) \in \mathbb{R}[z] \) of a degree not exceeding \( d \), that is,

\[
p_k(z) = \sum_{j=0}^{d} a_{k,j} z^j,
\]

is obtained by solving the set of normal equations. Particularly, gathering the \( d + 1 \) unknown polynomial coefficients into vector

\[
a_k = [a_{k,0}, a_{k,1}, a_{k,2}, \ldots, a_{k,d}]^T
\]

and the \( m \) sample values into vector

\[
x_k = [x_{k-l}, \ldots, x_k, \ldots, x_{k+l}]^T,
\]

one immediately gets the system of normal equations [70]

\[
J^T J a_k = J^T x_k,
\]

yielding the solution

\[
a_k = (J^T J)^{-1} J^T x_k,
\]

where the Jacobian matrix \( J \in \text{Hom}(\mathbb{R}^{d+1}, \mathbb{R}^m) \) is such that

\[
J_{ij} = (i - l - 1)^{j-1}, \quad i \in \{1, \ldots, m\}, \quad j \in \{1, \ldots, d + 1\},
\]

or, equivalently,

\[
J = \begin{pmatrix}
1 & -l & (-l)^2 & \ldots & (-l)^d \\
1 & -l+1 & (-l+1)^2 & \ldots & (-l+1)^d \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & l-1 & (l-1)^2 & \ldots & (l-1)^d \\
1 & l & l^2 & \ldots & l^d
\end{pmatrix}.
\]
2.3. POST-PROCESSING TECHNIQUES

One can also revise \[117\] the above process, stemming from the system of normal equations (2.6) and leading to the solution (2.7), as solving the (over-determined) linear system

\[ J a_k = x_k \] (2.10)

by means of the *Moore-Penrose pseudoinverse matrix* \( J^\dagger \) of \( J \) \[101, 116\]. If the matrix \((J^T J)^{-1}\) is non singular (as in the case of least squares, see the following theorem §2.1), then the Moore-Penrose pseudoinverse is explicitly given by

\[ J^\dagger := (J^T J)^{-1} J^T, \] (2.11)

where \( J^T \) is used in place of \( J^H \), because the entries of \( J \) matrix are real, so that the Hermitian transpose \( J^H \) coincides with the ordinary transpose \( J^T \). Thanks to the affinity (2.2), the pseudoinverse \( J^\dagger \) is independent of the particular \( k \)-th central sample and may thus be computed just once and for all, with obvious computational advantages. An accurate and stable way to numerically compute the pseudoinverse (2.11) is by means of the *singular value decomposition* (SVD) \[49\], which holds for arbitrary matrices over the real or the complex field. The singular value decomposition of matrix \( J \) reads \( J = U \Sigma V^T \). In the SVD, \( \Sigma \in \text{Hom}(\mathbb{R}^{d+1}, \mathbb{R}^m) \) is a diagonal (generally non square) matrix storing the singular values of \( J \) on its diagonal, and being otherwise null. The matrices \( U \in \text{Aut}(\mathbb{R}^m, \mathbb{R}^m) \) and \( V \in \text{Aut}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1}) \) collect the left- and right-singular vectors of \( J \), respectively, in correspondence with the singular values. Both matrices are unitary. Since \( J \) is real valued, so are also \( U \) and \( V \). Therefore, they are actually orthogonal and their Hermitian transpose coincide with their common transpose. Owing to the properties of the pseudoinverse matrix and of the SVD, one easily gets

\[ J^\dagger := V \Sigma^\dagger U^T, \] (2.12)

where \( \Sigma^\dagger \in \text{Hom}(\mathbb{R}^m, \mathbb{R}^{d+1}) \) is the pseudoinverse of \( \Sigma \), which is a diagonal (generally non square) matrix storing the inverse of non-zero singular values of \( J \) on its diagonal, and being otherwise null.

One obvious question arise as to the existence and uniqueness of the solution (2.7) to problem (2.6), which would imply the possibility to always apply Savitsky-Golay filtering, without obstacles, to any time series, as already implicitly argued when mentioning the SVD. The answer, positive, is given by the following

**Theorem 2.1.** The matrix \( J^T J \) is non singular, i.e., \( \det J^T J \neq 0 \), so that its inverse exists and the best fit local polynomial may always be computed. Moreover, \( J^T J \) has full rank, so that the best fit polynomial is unique.

**Proof.** Matrix \( J \in \text{Hom}(\mathbb{R}^{d+1}, \mathbb{R}^m) \) and thus \( J^T J \in \text{End}(\mathbb{R}^{d+1}) \). First, \( J \) has exactly its maximum rank \( d + 1 \) (remember that \( m > d \), i.e., \( m \geq d + 1 \)). The proof is by contradiction.
If this was not the case, then the columns of matrix (2.9) would be linearly dependent, and then there would exist a set of real coefficients \( \alpha_j \in \mathbb{R} \), \( j \in \{0, \ldots, d\} \), not all null, such that

\[
\sum_{j=0}^{d} \alpha_j z^j = 0, \quad z \in \{-l, \ldots, 0, \ldots, l\}.
\]

Then the polynomial

\[
q(z) := \sum_{j=0}^{d} \alpha_j z^j \in \mathbb{R}[z]
\]

would have at least \( m > d = \deg(q) \) roots over the complex field \( \mathbb{C} \) (actually, over the ring of integers \( \mathbb{Z} \)), which is obviously absurd, due to the fundamental theorem of Algebra. To complete the proof, one notices that, for a general matrix \( A \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \), \( n, m \in \mathbb{Z} \), it is always \( \ker A^T A = \ker A \). For, if \( x \in \ker A^T A \), then \( A^T A x = 0 \) and consequently \( x^T A^T A x = 0 \), so that \( \|Ax\|_2 = 0 \) and, by the norm defining properties, \( Ax = 0 \). Thus \( x \in \ker A \) as well, and \( \ker A^T A \subseteq \ker A \). The opposite inclusion \( \ker A \subseteq \ker A^T A \) follows trivially because \( Ax = 0 \) implies \( A^T A x = A^T 0 = 0 \). Thus \( \dim \ker A = \dim \ker A^T A \).

Since the map produced by the homomorphism \( A \) is clearly surjective over its own image and considering the canonical projection over the quotient vector subspace, the following diagram

\[
\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{A} & \text{im} A \\
\downarrow & & \downarrow \\
\mathbb{R}^n & \xrightarrow{\ker A} & \ker A
\end{array}
\]

is commutative and the first isomorphism theorem for vector spaces yields

\[
\mathbb{R}^n / \ker A \cong \text{im} A,
\]

where \( \cong \) stands for vector space isomorphism. Thus \( \mathbb{R}^n = \ker A \oplus \text{im} A \). Likewise, \( \mathbb{R}^n = \ker A^T A \oplus \text{im} A^T A \), so that \( \dim \ker A + \dim \text{im} A = n = \dim \ker A^T A + \dim \text{im} A^T A \). Finally, since the two kernels have the same dimension, then \( \dim \text{im} A = \dim \text{im} A^T A \), that means, the two matrices \( A \) and \( A^T A \) have the same rank (this is essentially a trivial application of the rank-nullity theorem, where \( \text{nullity} := \dim \ker \) and \( \text{rank} := \dim \text{im} \)). As a special case, the two matrices \( J^T J \) and \( J \) have the same rank \( d + 1 \) and, for Rouché-Capelli’s theorem, \( J^T J \in \text{Aut}(\mathbb{R}^{d+1}) \) and therefore it is non-singular. Then the system of normal equations always admits one, and just one, solution.

By naming \( c_n, n \in \{-l, \ldots, l\} \), the entries on the first row of the pseudoinverse \( J^T \), which does not depend on the signal to be filtered, the \( k \)-th sample of the filtered signal is seen to be obtained from

\[
y_k = \sum_{n=-l}^{l} c_n x_{k+n}, \quad \forall k \in K.
\]

Hence Savitsky-Golay is a linear filter. Working similarly on subsequent rows, analogous expressions may be easily found for the time series of derivatives as well.
2.3. POST-PROCESSING TECHNIQUES

Figure 2.15: Arc voltage, arc current and arc conductance in case of a successful interruption.

Figure 2.16: Post-arc current and conductance (bold lines denote filtered signals, thin lines raw signals).
Additionally, one could filter, as a special case, the constant time series \( \{(t_k, 1)\}_{k \in K} \) identically equal to 1, that is, \( x_k \equiv 1, \forall k \in K \). One notices that \( a_{k,0} = 1 \) (that is, the output signal is valued 1) and \( a_{k,j} = 0 \) for \( j \geq 1 \) (that is, all derivatives vanish) is an obvious solution to the normal equations \( \forall k \in K \). Then it is the solution, which we know be unique. This simply means that the signal is already (trivially) smooth and needs not further smoothing, with the output time series equal to the input time series. Thus
\[
\sum_{n=-l}^{l} c_n = 1.
\]
Hence Savitsky-Golay filter is a moving average (of a rather elaborated kind).

The Savitsky-Golay method acts as a low pass filter, with a cutting frequency related to the amplitude of the moving window (the wider the window, the lower the cutting frequency). It is better to keep the degree of the polynomial low, e.g., \( d = 3 \). The filter may be applied in more then one pass, in order to increase smoothing. One main virtue of the filter is being local, that is, it does not require to operate on the whole set \( K \) of samples to output one single smoothed sample, unlike, e.g., Fourier transform based filters.

After polynomial (2.3) is found, the smoothed sample is given by
\[
y_k = p_k(0) = a_{k,0}
\] (2.13)
and, according to (2.2), the time derivative by
\[
\dot{y}_k = \frac{d}{dt} p_k(z(t)) = \frac{dp_k}{dz} \bigg|_{z=0} \frac{dz}{dt} = \frac{a_{k,1}}{h}.
\] (2.14)
Derivatives up to order \( d \) may be likewise computed. By induction, the general \( \ell \)-th derivative \( y_k^{(\ell)} \) is readily found to be
\[
y_k^{(\ell)} = \ell! \frac{a_{k,\ell}}{h^\ell}, \quad (\ell \leq d).
\] (2.15)

In the analysis of electrical test oscillograms, the Savitsky-Golay filter is applied to both arc voltage drop and arc current time histories \( \{t_k, u(t_k)\}_{k \in K} \) and \( \{t_k, i(t_k)\}_{k \in K} \), respectively. Together with the filtered versions of the latter two signals, their time derivatives are also computed. Then the time derivative of arc conductance \( g = i/u \) is conveniently computed as
\[
\frac{dg}{dt} = \frac{1}{u^2} \left( \frac{di}{dt} \frac{1}{u} - i \frac{du}{dt} \right).
\] (2.16)
This way, a numerically stable signal of conductance time derivative \( \{t_k, g(t_k)\}_{k \in K} \) may be deduced from stable estimates of current, voltage and their first derivatives.
2.3. POST-PROCESSING TECHNIQUES

2.3.3 Current Zero and Voltage Zero Synchronization

Care must be taken in handling the current zero, since it contains the crucial information to evaluate the interruption. This implies spotting the current and voltage zeros and synchronizing the two signals in such a way that the two instants coincide.

Each filtered signal, i.e., arc current and voltage, is a real valued sequence \( \{t_k, x_k\}_{k \in \mathcal{K}} \). Additionally, the signal may also be assumed continuous. We consider any closed and bounded time interval \([t_{k_1}, t_{k_2}] \subset \mathbb{R}\) which, for Heine-Borel lemma [56], is compact. For Bolzano’s theorem [3], if a continuous, real function over a compact set is sometimes positive and sometimes negative, then it must be zero at some point of the compact. From the physics of current zero we know in advance that both current and voltage must be crossing the zero value, and therefore we can exclude zeros with even multiplicity. It follows that an exhaustive search of the zeros of the sequence \( \{t_k, x_k\}_{k \in \mathcal{K}} \) can be carried out by letting \( k_1 \) span \( \mathcal{K} \), fixing \( k_2 = k_1 + 1 \) and checking whether

\[
x_{k_1} \cdot x_{k_2} \leq 0,
\]

which, for what above, implies that a zero must exist for some \( t_0 \in [t_{k_1}, t_{k_2}] \).

The exact zero instant \( t_0 \) is estimated by linear interpolation in between \( t_{k_1} \) and \( t_{k_2} \) as

\[
t_0 = t_{k_1} - x_{k_1} \cdot \frac{t_{k_2} - t_{k_1}}{x_{k_2} - x_{k_1}}.
\]

In general, the zero instant for measured current will be different from that for measured voltage. Since the electric arc must be very well assimilated to a (nonlinear) ohmic component, from a physical standpoint we expect the two signals to assume the zero value simultaneously. Current and voltage zero synchronization is carried out by shifting “horizontally”, i.e., in time, the current signal by an amount equal to the (small) inescapable difference between the two zeros. In general, the horizontal shift is different from the sampling interval, so that the shifted signal must be re-sampled in order to allow a unique sequence of time instants to be valid both for current and voltage. Re-sampling is carried out by finding through linear interpolation the new values of the signals required. The choice to shift the current signal so that its zero coincide with that of the voltage signal and not vice versa is arbitrary. We prefer doing so because the voltage signal allows a very sharp detection of its zero.
2.4 Optical Measurements

Besides electrical measurements, the behavior of circuit breakers may be investigated by means of optical methods. The latter category mainly includes filming with high speed camera and fiber optics. As regards high speed cameras, the main drawback lies in the necessity of substituting a portion or all of the breaker external cover with some transparent material, which could alter the arc evolution, due to different gassing properties of the materials employed. An additional problem is the difficult trade-off between temporal sampling and spatial resolution.

In this context, fiber optics can be seen as a less invasive technique, since only thin holes need to be drilled into the breaker sidewall, and endowed with superabundant sampling rate, as later detailed. Other than arc motion in circuit breakers, there are obviously many engineering applications where the medium of interest is a gas at high temperature in an unsteady flow regime. Typical examples include combustion, such as in internal combustion engines. High-speed imaging system have made it possible to study phenomena that are not revealed to the naked eye. Such imaging systems can provide qualitative and also, in many cases, semi-quantitative or quantitative information on the processes being studied.

In the next sections we will show some results obtained thanks to the collaborations with TaiCaan Technologies\(^1\), which have led to the development of an Arc Imaging System (AIS) for arc visualization during the short-circuit. This system, briefly described below, provides a parallel acquisition of the light emitted by the arc through (up to) 96 optical fibers, looking inside the arc chamber; all the digital signals are then post-processed with a software we developed autonomously in order to create a movie of the arc inside the chamber. The same software is also able to extract valuable quantitative information such as the arc root trajectory along the fixed contact. A simple example is shown here, but it is obvious that this kind of investigation may shed light on the influence of various geometric variables on the arc movement. The concepts outlined in this section have been published in a journal paper [90].

2.4.1 Experimental Setup

Optical measures are carried out in the test lab on circuit breakers undergoing the same tests described when dealing with electrical measurements. Fibers are mounted on one side of the circuit breaker under test, as shown in Figure 2.17, displaying a picture of an ABB T3 low voltage circuit breaker, equipped

\(^1\)www.taicaan.com
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Figure 2.17: An ABB T3 low voltage circuit breaker equipped with optical fibers before a test.

Figure 2.18: Assembly of optical fibers, with interposed protection glass, onto an ABB T3 low voltage circuit breaker.
with 51 optical fibers before a test. In order to allow light to exit from the arc chamber, one hole for each fiber used in the experiment is drilled in the plastic sidewall of the breaker, as shown in Figure 2.18. In the same figure it is also visible a quartz glass, used to protect the fiber from the hot gas, and an additional drilled plate, termed the optical head material, which strengthens the fiber bundle.

The position and diameter of the holes in the sidewall of the breaker are a delicate issue during the design of the test and setup, since the inner geometry of the arc chamber is a serious geometric constraint. Particularly, a hole cannot be drilled in those places where, in the inner side, there is a splitter plate, a screw or the fixed contact. The preparation of a breaker for optical measurements is thus a time consuming activity and such kind of tests can hardly be envisaged to become standard routine along with electrical tests.

The distance of the end of each fiber from the arc chamber, where the light is emitted, and the diameter of the holes in the optical head material (see Figure 2.19) are also very important to increase the quality of the final result. If the aforementioned distance is increased, in fact, each fiber will become more and more selective, i.e., it will transmit only the light originated just in front of it. This is a desirable result, but it is counterbalanced by sensible attenuation if the fiber is too distant.

A calibration of each fiber is also needed in order to get rid of inevitable
Figure 2.20: Signal attenuation vs. wavelength.

2.4. OPTICAL MEASUREMENTS

2.4.2 Arc Imaging System (AIS)

The AIS system is a self-contained mobile system, which can be supplied with a range of optical fiber lengths. The portable system includes a purpose built PC and an integrated computer screen and keyboard (KVM). The hardware is mounted in vibration proof mountings and can be sealed for transportation. Polymer fibers with a 1 mm core diameter and 0.6 mm sheath (2.2 mm overall diameter) are used. Signal attenuation in polymer fibers is typically $-0.2 \, \text{dB/m}$ at 665 nm light wavelength, and $-1.5 \, \text{dB/m}$ at 820 nm wavelength. The full graph of signal attenuation over a spectrum inclusive of visible light (from about 380 nm to 750 nm [145]) and part of infrared radiation (from 750 nm to 1000 nm) is shown in Figure 2.20. The less attenuated wave band is seen to coincide with green (495 nm and 570 nm) and yellow (570 nm and 590 nm) light. The fiber is suited to relatively small lengths of 1 − 100 m. In the system presented the fiber length is 5 m allowing electrical insulation between
the arcing event and the data acquisition. The main advantages of polymer fibers lie in its ease of manipulation and an aperture comparable with the resolution required.

The AIS settings can be fine-tuned for different applications. This can be accomplished by alteration of the following parameters:

- The **gain** of the system, which can be set between $\times 1$ and $\times 32$. Several tests are needed to set this parameter properly, as the maximum light level of the arc is not known in advance and, above all, its dependence from the testing conditions is uncertain. Nonetheless, a correct individualization of the expected light intensity of the event is crucial to visualize the whole phenomenon and prevent from fiber saturation. For all the tested prospective currents, ranging from 500 $A$ to 5000 $A$, the best gain always proved to be $\times 16$. Therefore, there is evidence that the light intensity peak is independent of the current, at least in the tested range. The hot region, where the light intensity approaches its peak is wider for higher currents. This is in agreement with radiation theory (see §3.8), according to which the maximum temperature in a plasma depends on its chemical composition.

- The **sampling rate**, which can be up to $1 MHz$. From the Nyquist-Shannon sampling theorem [138], the sampling rate must be at least twice

![Figure 2.21: A typical arc voltage time history, from which it is observed that a sampling rate of 100 $kHz$, or higher, is accurate enough to resolve the arc motion.](image-url)
as high as the maximum frequency of the processes we are interested in. In the case at hand, the physical process under investigation is represented
by the sudden arc movements between the arc chamber and the contacts,
whose effects are easily found in arc voltage fluctuations. Therefore, it is
easy to determine the required sampling rate by inspection of arc voltage
time histories. A typical example is shown in Figure 2.21, where it is
observed that it takes roughly $20\,\mu s$ for the arc to change its position.
Therefore, a sampling rate of $100\,kHz$, or higher, is accurate enough to
resolve the arc motion.

- The number of fibers used. The system hosts a total of 6 cards, each
  one with 16 channels for data acquisition, for a total of 96 channels. The
  number of channels is a consequence of what one wants to investigate.
  For instance, measuring the movement of the arc roots requires less fibers
  than studying the movement of the whole plasma column inside the arc
  chamber. On the other hand, the size of the circuit breaker to be investig-
  ated poses strong dimensional constraints on the number of fibers which
  can be used.

The direct output of the system is a collection of 8 bit signals, one per fiber
(see, e.g., Figure 2.22). Each signal represents the light intensity time history
captured in a particular spot of the circuit breaker under test. Since we have a
resolution of 8 bits, a maximum of $2^8 = 256$ levels are available. The digital
information coming from each fiber is recorded in a RAM with storage for
512 $kb$ values per channel. Consequently, the $1\,MHz$ maximum sampling rate

![Figure 2.22: Sample output light intensity signal from a fiber.](image-url)
allows recording over 0.5 s of data capture. Coarser time resolutions would yield longer recording times, but this is seen to be unnecessary.

We developed an automatic routine in order to produce movies of the arc evolution, starting from the light intensity signals. The routine is called off-line, after the test has been performed. For any sampled time instant, the light intensity value from each fiber is mapped onto a suitable color scale and superimposed to a picture of the circuit breaker arc chamber, in the correct relevant location. The fiber spots are then interpreted as the nodes of a planar triangulation, which is automatically produced by means of the classical Delaunay algorithm [33]. The light intensity in the points of each triangle of the triangulation is estimated by linear interpolation and is then mapped to the color scale accordingly. Particularly, if \( V_j, j \in \{1, 2, 3\} \), are the vertices of a triangle \( T \) containing a point \( P \), and if \( I_j, j \in \{1, 2, 3\} \), are the light intensity values in the three vertices, respectively, then the light intensity value in point \( P \) is

\[
I = \frac{1}{A} \sum_{j=1}^{3} I_j A_j,
\]

where \( A_j \) is the area of the sub-triangle with vertices \( P \) and the two vertices of \( T \) other than \( V_j \), and \( A = \sum_{j=1}^{3} A_j \) is the total area of \( T \). It is immediately seen that, when \( P \) lies onto some edge of \( T \), then the classical 1D linear interpolation scheme over a line segment is attained, and if \( P = V_j \), for some \( j \), then \( I = I_j \). According to this scheme, all of the points belonging to the convex hull of the collection of fiber spots are given a suitable color.

The linear interpolation assumption is obviously arbitrary and care must be paid when examining and interpreting the visual result, since any interpolation scheme is unsuitable whenever some obstacle, such as the solid portion of a splitter plate, is interposed between two fiber spots. The mobile contact position cannot be monitored with precision and it is estimated so to match the light intensity distribution. Frames produced time wise are then mounted to produce a movie.

The light intensity signals from a selected number of fibers, precisely those located along the fixed contact and the arc runner, may be further processed to monitor the arc root motion. We follow the center of intensity (COI) technique, as introduced by McBride et al. [164, 93, 91, 92]. Precisely, the arc root coordinates \((x_{\text{root}}(t), y_{\text{root}}(t))\) at sampled time \( t \) are computed as the centroid of a discrete distribution, that is,

\[
x_{\text{root}}(t) = \frac{\sum_{j \in J_{\text{root}}} x_j I_j(t)}{\sum_{j \in J_{\text{root}}} I_j(t)}, \quad y_{\text{root}}(t) = \frac{\sum_{j \in J_{\text{root}}} y_j I_j(t)}{\sum_{j \in J_{\text{root}}} I_j(t)},
\]
where $J_{\text{root}}$ is a suitable index set referencing the fibers selected as above and $(x_j, y_j)$ is the location of the $j$-th fiber, with light intensity $I_j$. The arc root position $s_{\text{root}}$ is computed as the distance from a reference fiber $(x_{\text{ref}}, y_{\text{ref}})$, located at one extreme point of the possible arc root path, that is,

$$s_{\text{root}}(t) = \sqrt{(x_{\text{root}}(t) - x_{\text{ref}})^2 + (y_{\text{root}}(t) - y_{\text{ref}})^2}.$$  

### 2.4.3 Results

Optical inspection may provide very useful information to understand the arc evolution and to have an empirical confirmation of some properties that can be predicted from a theoretical standpoint.

Results need being correctly interpreted. First, some fiber can be progressively blinded by the soot or by the liquid metal droplets, produced during the arc event and deposited onto the internal side of the interposed protection glass. Second, it must always be remembered that the output signal is the light produced by the arc, and not the arc current density map. Even though the two concepts are interrelated, they do not coincide exactly. Current density produces high plasma temperatures, due to Joule heating, and current flows in hot (and thus conductive) plasma. Also, light is the radiative energy emitted by hot plasma. Nonetheless, plasma is a participating medium (see §3.8), interacting with radiation. A light beam is scattered along its path, and it is partially absorbed by the trespassed gas molecules, which are consequently heated up and emit radiation in their turn. Light intensity filtering and light frequency filtering here play a crucial role, since, by Planck’s law, the peak of emitted radiative heat is source-temperature dependent.

**Test I.** We hereafter report the results of a first test on an ABB T3 low voltage circuit breaker. The network parameters have been adjusted in order to feed the left pole of the breaker with a $400\ V$ voltage supply and a $5.8\ kA$ prospective current. The $\cos\varphi$ has been set to 0.54. These values have been chosen as they do not represent a severe testing condition for the breaker, neither from a structural nor interruption point of view.

Eight frames are shown in Figure 2.23, while in Figure 2.24 the correspondence with the oscillogram can be found. The arc is ignited in frame 1 and a sudden but relatively small voltage jump may be observed in correspondence. The voltage increment is due to the creation of a cathodic and an anodic arc root spots. The arc elongates from frame 2 to frame 6 and moves into the arc chamber. Frame 6 shows only a small fraction of the arc since the current is decreasing rapidly, and only the hottest part of the plasma is visible. In this experiment, the mobile contact played the role of the cathode, which is known
Figure 2.23: Arc evolution during a short-circuit as observed with fiber optics (test 1).
to be considerably hotter than the anode (see §3.9.3). This can be appreciated rather well in the last frames, relevant to a time period when the arc is cooler and thus the differences in light emission in the two arc roots are not covered by saturation.

In frame 7 a sudden drop in the arc voltage is observed. The AIS clearly shows that this happens in correspondence with a back re-ignition of the arc in the central region of the chamber. These kinds of phenomena are extremely dangerous if they occur just before the current zero, since they put the basis for a thermal re-ignition which may lead to another semi-cycle of arcing. A physical explanation of the phenomenon is related to the chaotic fluctuations of the arc plasma in the chamber, and particularly with or without crossing the splitter plates. When most of the arc plasma cloud is crossing the splitter plates, that is, when it occupies the upper most third of the frame (like in frame 6, even though the light intensity level is low to appreciate), then many cathodic and anodic arc root spots are activated, each one providing an additional resistance (and voltage drop) to the arc. Vice versa, when the arc plasma cloud is mostly outside the splitter plates (like in frame 7), its resistance is only due to just one
cathodic and one anodic arc root spot, apart from the distributed resistivity of the plasma column, which is reasonably similar to the other case. Finally, the definitive extinction of the arc is observed in frame 8.

The position of the arc root attached to the fixed contact is shown vs. time in Figure 2.25. After the arc is ignited between the fixed and mobile contact, it stays quite still for a certain time (see frame 1 and 2 of Figure 2.23), which is extremely harmful for the plate and, in general, for the whole fixed contact, due to material ablation issues. This behavior is not completely understood, but it is probably due to the fact that the arc root needs to heat the path in front of it in order to generate a sufficient thermionic emission, which is crucial to self-sustain the cathode region [91, 92]. This period is called in the technical literature arc *immobility time*, and the study of the factors that can reduce it is an active field of research (see, e.g., [91, 92]), since the life of a breaker is strongly related to the arc ablation of the fixed contact during arc interruption. Once the metal is sufficiently hot, the arc moves forward. This sudden change
in the speed of the arc root is easily seen in Figure 2.25 (see frame 4 of Figure 2.23) and approximately occurs slightly before 4 ms.

**Test II.** Figure 2.26 shows another example of a short-circuit in which the thermal re-ignition has been caused by an increased test current and voltage. In particular, the test has been done on another ABB breaker, at 4kA@726V and with a power factor of 0.5. Frame number 1, which corresponds to point 1 in the oscillogram showed in Figure 2.27, grabs the arc ignition due to contact separation, while frame 2 shows the slow arc elongation driven by the contact movement.

Frame 3 shows a further - and, as a matter of fact, welcome - elongation of the arc inside the arc chamber, which unfortunately happens only just before current zero. In this point, the arc is forced to a momentary extinction where electron-ion recombination begins to take place and the transient recovery voltage, i.e., the negative spike in the voltage signal after current zero, starts to develop across the breaker. The test is beyond the rated interrupting capability of this breaker, so a re-ignition follows, as shown in frame 4. This frame is of great interest, as it grabs the re-ignition spot, which is of course not evident by visual inspection after the test. Precisely, on the fixed contact side, the arc root is located in the upper-left part of the frame, well above the fixed contact plate and completely inside the arc chamber region. On the mobile contact side, the arc root is located on the plate. Since the main purpose of a breaker is to interrupt the current at the first zero, the position of the re-ignition spot is valuable information for the designers in order to accurately study this region and understand why the re-ignition has occurred.

After re-ignition, magnetic forces push the arc inside the chamber, see frame 5. This situation, anyway, is not stable. In fact, as the arc enters the splitter plates, its temperature and conductance drop until an alternative path, i.e., a more conductive one, is found. This corresponds to frame 6, where the arc finds its way near the plate of the mobile contact and its voltage is at a lower peak. Possibly, the newly formed arc root is not hot enough to stabilize the arc in this position, so the magnetic forces manage to push the arc inside the chamber again; see frame 7, which corresponds to a higher peak in the voltage. The arc oscillates with an increasing frequency, up to 1 MHz, in between lower and upper peaks of voltage. Lower (upper) voltage peaks are found when the arc is mostly outside (inside) the splitter plate rack, as expected. During this oscillatory arc motion, the arc root on the fixed contact side is nearly pinned well above the relevant plate, whereas the arc root on the mobile contact side also oscillates back and forth in between the relevant plate and the uppermost point of the mobile contact. After this unstable phase, the arc stabilizes around the mobile contact and the voltage looks flat. This, of course, is an undesirable situation, which leads to a high arc conductivity and a re-ignition at the next
Figure 2.26: Arc evolution during a short-circuit as observed with fiber optics (test II).
2.5 Performance Evaluators

In order to evaluate the breaker’s interrupting performance, we define a set of evaluators

\[ \eta_k : L^2(\mathbb{R}) \rightarrow \mathbb{R}^+, \]

i.e., non-negative, real valued functionals to be applied to the time histories of electrical quantities in the vicinity of any current zero. We look for evaluators whose value is correlated with the successfulness of the interruption. The concepts outlined in this section have been published in a journal paper [9].

Figure 2.27: Arc voltage and arc current vs. time (test II).
2.5.1 Macroscopic Evaluators

An evaluator will be termed *macroscopic* if it requires the value of one or more quantities over a time span whose amplitude is comparable with half the AC period, i.e., 10 ms. Considering the time period from the beginning of arcing (conventionally set as the origin of the time line) to the first current zero $t_0$, we define:

1. The arc energy
   \[ \eta_1 := \int_0^{t_0} u(t) \cdot i(t) \, dt; \] (2.17)

2. The mean value of the arc current
   \[ \eta_2 := \frac{1}{t_0} \int_0^{t_0} |i(t)| \, dt; \] (2.18)

3. The maximum value of the arc current
   \[ \eta_3 := \max_{t \in [0,t_0]} |i(t)|; \] (2.19)

4. The mean value of the arc voltage
   \[ \eta_4 := \frac{1}{t_0} \int_0^{t_0} |u(t)| \, dt; \] (2.20)

5. The maximum value of the arc voltage
   \[ \eta_5 := \max_{t \in [0,t_0]} |u(t)|. \] (2.21)

Standard test lab instrumentation (see item 6 and 8 in Figure 2.1) can be used for macroscopic evaluators, because only global time histories are involved and the post-arc region needs not being recorded in detail. Straightforward signal processing can be applied, a decent noise removal being only required to prevent the max operator from returning a fictitious spike.

Macroscopic evaluators computed from experimental tests are collected in Figure 2.28, where white and black squares denote successful interruptions and failures, respect. Only the first shot on each pole of the breaker is here considered. The absence of a good ordering based on the interruption quality may be clearly observed, so that none of the macroscopic evaluators is able to predict the outcome of the test. This picture confirms the idea according to which the fate of a breaker in terms of arc interruption is decided in the
Figure 2.28: Macroscopic interruption evaluators. Left to right: arc energy [J], mean arc current [A], max arc current [A], mean arc voltage [V], max arc voltage [V]. White squares denote successes, black denote failures.
time interval around current zero, where ion recombination processes and non
equilibrium physics dominate. Roughly, the time constant of these processes is
in the order of tenths of microseconds, and macroscopic evaluators are mostly
based on data too far away from the zone of interest to be useful. In conclusion,
although macroscopic evaluators can be simply obtained, they are not very
useful for evaluating interrupting performances.

2.5.2 Microscopic Evaluators

An evaluator will be termed *microscopic* if it only requires the value of one or
more quantities over a very short time span with reference to the period of AC
current, such as hundreds of microseconds, or even a single instant in time. We
consider the period close to current zero, since many authors [167, 14, 144] have
shown that this is a suitable area to evaluate the interrupting performance. We
use a time span $T = 10\mu s$ after each current zero and define:

1. The current slope precisely at current zero

   $\eta_6 := \left| \frac{di}{dt} \right|_{t=t_0}$; \hfill (2.22)

2. The electric charge passing through the post-arc channel over the time
interval $T$, i.e., $L^1$ norm of arc current over the interval $(t_0, t_0 + T)$,

   $\eta_7 := \int_{t_0}^{t_0+T} |i(t)| \, dt$; \hfill (2.23)

3. The Joule’s integral\(^2\) over the time interval $T$, i.e., $L^2$ norm of arc current
over the interval $(t_0, t_0 + T)$,

   $\eta_8 := \int_{t_0}^{t_0+T} i^2(t) \, dt$; \hfill (2.24)

4. The conductance at current zero

   $\eta_9 := g(t_0)$; \hfill (2.25)

\(^2\)Sometimes, in the technical community, such a quantity is improperly termed *energy.*
Despite this being always wrong also for ohmic components, since energy would be the time
integral of $ui = i^2/g$ and not of $i^2$, for constant resistances the denomination is up to a
multiplicative factor. The mistake is catastrophically wrong and misleading in case of time
variable resistors, such as the electric arc.
5. The average current slope before current zero

\[ \eta_{10} := \frac{|i(t_{10})| - |i(t_{50})|}{t_{10} - t_{50}}, \]  

(2.26)

where \( t_{10} = t_0 - 10\mu s \) and \( t_{50} = t_0 - 50\mu s \).

Except for \( \eta_{10} \), a sensitive current measuring device (see item 10 in Figure 2.1) is required for microscopic evaluators, due to small currents in their definition. A good signal processing algorithm, however, is also needed, and Savitsky-Golay method is particularly welcome for \( \eta_6 \), where a precise evaluation of a derivative is also required. In the case of \( \eta_7 \) and \( \eta_8 \), the integral tends to compensate for high frequency fluctuations, so that filtering becomes less important. On the contrary, an accurate zero crossing detection is required for the first four evaluators.

Microscopic evaluators computed on experimental tests are collected in Figure 2.29, where white and black squares denote successful interruptions and failures, respectively. Only the first shot on each pole of the breaker is here considered. In contrast with the macroscopic case, the first four evaluators are a good way to judge the “severity” of a test, because a good ordering can be found with a clear threshold between interruptions and failures. This is the essential prerogative every good evaluator should possess. As regards indicators \( \eta_7 \) and \( \eta_8 \), we stress that the duration \( T \) of the interval over which they are computed is very small compared to the characteristic time for an arc to restrike, i.e., hundreds of microseconds. Obviously, if \( T \) was as long as the time for current to recover from very small values to those of a failed interruption, then both the \( L^1 \) norm and the \( L^2 \) norm of current would be very small for successful interruptions and relatively very big for failures, so that the discriminating power of these indicators would be trivial and useless.

Rather surprisingly, the average current slope before current zero seems to be completely unrelated to the interruption performance of the breaker, in contrast with what often has been observed for medium- and high-voltage breakers [131]. This result is a peculiarity of low voltage breakers, and can be explained with a careful analysis of the interaction between the arc and the supply network. As can be observed from Figure 2.15, the current descent towards current zero may be very well regarded as being linear for the testing conditions. The network equation (2.1), in which the contribution of the arc voltage \( u \) is considered, now reads

\[ L \frac{di}{dt} + Ri + u = V_s \sin(\omega t + \theta), \]  

(2.27)

where the phase shift \( \theta \) is such to be coherent with the (arbitrary) choice to locate the origin of time in a suitable instant (slightly before the current zero
Figure 2.29: Microscopic interruption evaluators. Left to right: current slope at CZ [A/s], electric charge through the post-arc channel [C], Joule integral [A^2s], conductance at CZ [S], average current slope before CZ [A/s]. White squares denote successes, black denote failures.
2.5. PERFORMANCE EVALUATORS

Figure 2.30: The supply voltage has, at current zero, a value which depends on the position of the current zero itself.

at hand, as later explained). As a consequence, its value is different from the one appearing in equation (2.9), where the origin of time is usually located at the beginning of the short-circuit. Solving equation (2.27) for a certain instant $t^*$ sufficiently close to current zero (say 50 $\mu$s before), so that ohmic voltage drop across the series resistor in the supply circuit becomes negligible, one gets

$$\dot{I}^* \approx \frac{V_s \sin(\omega t^* + \theta) - u^*}{L}, \tag{2.28}$$

where quantities with an asterisk are evaluated at $t^*$, and $u^*$, $I^*$, $V_s$ and $\omega$ are arc voltage drop, arc current, supply voltage amplitude and radial frequency, respectively.

The inductance can be computed from the imaginary part of the impedance and reads

$$L = \frac{V_s}{\omega I_p \sin \varphi}, \tag{2.29}$$

$I_p$ being the prospective current. If $t^*$ is sufficiently close to current zero, from (2.28) we see that the average current slope before zero is related to its value exactly at the zero crossing by

$$\dot{I}^* \approx \dot{I}_0 - \frac{\omega I_p}{\sin \varphi} u^*. \tag{2.30}$$

From this equation we deduce that, if $V_s \gg u^*$, then

$$\dot{I}^* \approx \dot{I}_0, \tag{2.31}$$

so that the two stemming evaluators are equivalent. This is not true in the LV realm, where supply and arc voltage are of the same order of magnitude; in other words, there's no reason to expect that the current slope evaluated before current zero and precisely at the current zero carry the same information. As a matter of fact, current slope noticeably decreases before reaching the current zero (see §2.2.9).
A somewhat deeper insight can be gained from classical black box modeling, such as Mayr’s one \cite{88} (see §5.3.1), where the arc conductance $g(t)$ is related to arc voltage $u(t)$ and arc current $i(t)$ by the differential equation (5.29), that reads

$$\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau} \left( \frac{ui}{P_{out}} - 1 \right),$$

where $P_{out}$ and $\tau$ are two constant model parameters $\in \mathbb{R}^+$. There is experimental evidence that the arc-network interaction is such that arc current descends practically linearly toward the current zero, before bending very close to it. By suitably setting the origin of time in the crossing point of the linear descent line, one may thus assume the expression $i(t) = I^* t$. Given such a linear expression for arc current, one may also remove arc conductance by substituting the l.h.s. of Mayr model according to (5.42), yielding

$$\frac{du}{dt} - \left( \frac{1}{t} + \frac{1}{\tau} \right) u + \frac{I^* u^2}{\tau P_{out}} t = 0$$

after some simple algebraic re-arrangements. Setting $z(t) := 1/u(t)$, this equation takes the form of the first order, linear, ordinary differential equation

$$\frac{dz}{dt} = - \left( \frac{1}{t} + \frac{1}{\tau} \right) z + \frac{I^*}{\tau P_{out}} t.$$

By setting $p(t) := -(1/t + 1/\tau)$ and $q(t) := I^* t / (\tau P_{out})$, one gets the standard form

$$\frac{dz}{dt} = p(t) z + q(t).$$

Despite the variability of the coefficients, the general solution (Leibniz, 1696, and Johann Bernoulli, 1697) is readily found and reads \cite{22}

$$z(t) = e^{\int p(t) dt} \left(C + \int q(t) e^{-\int p(t) dt} dt\right).$$

After simple computations, one finds the expression of arc voltage to be

$$u(t) = \frac{t}{Ce^{-\frac{t}{\tau}} + \frac{I^*}{P_{out}} (t^2 - 2\tau t + 2\tau^2)}. \quad (2.32)$$

The constant $C$ in equation (2.32) has to be determined by means of one initial condition. Nonetheless, the exponentially decreasing factor $\exp(-t/\tau)$ quickly damps out the effects of such initial condition, as shown in Figure 2.31, where the arc behavior is simulated starting from different initial values of voltage. The physical interpretation of this fact is extremely interesting,
because this means that the exact value of the initial arc voltage is practically unimportant provided that the arc time constant $\tau$ is small with reference to the time duration over which one can think to use black box models, like Mayr’s or others, in order to describe the arc evolution from a purely electrical standpoint. The benefit from such a quick vanishing of initial conditions is well understood when examining the past history of the arc, which is usually very chaotically fluctuating and thus very difficult to predict. This doesn’t mean that the past history can be neglected *tout court*. In fact, the value of the (constant) current slope $\dot{I}^*$ depends on the voltage supply which, in turn, depends on the instant of zero crossing. Due to the arc-network interaction, this event is strongly related to the previous arc history and has to be calculated with other techniques.

Some more insight can be obtained by merging the results contained in equation (2.28) with those in equation (2.32). Particularly, the have the following

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2_31.png}
\caption{Vanishing effect of the initial conditions on the arc evolution.}
\end{figure}
Proposition 2.1. The average slope of current before current zero is given by

\[ |\dot{I}^*| = \frac{1}{2} \left( |a|I_p + \sqrt{(aI_p)^2 + b\frac{P_{out}I_p}{\tau V_s}} \right), \tag{2.33} \]

where

\[ a := \frac{\omega \sin(-\sqrt{2}\omega \tau + \theta)}{\sin \varphi}, \quad b := \frac{4\omega}{2(\sqrt{2} + 1) \sin \varphi}. \tag{2.34} \]

Proof. We can decide to choose as starting time \( t^* \) the one relevant the latest voltage peak before the current zero. Neglecting the effects of initial conditions in (2.32), i.e., setting \( C = 0 \), the peak arc voltage \( u^* \) is easily calculated by considering the roots of the fist derivative. With simple computations one finds

\[ u^* = -\frac{1}{2(\sqrt{2} + 1)} \frac{P_{out}}{I^* \tau}, \tag{2.35} \]

occurring at \( t^* = -\sqrt{2}\tau \) (and thus before the zero crossing). Equation (2.28) can be evaluated precisely in \( t^* \), when the arc voltage reaches its last peak before the current zero. After straightforward manipulations one gets

\[ \dot{I}^* = \frac{1}{2} \left( aI_p \pm \sqrt{(aI_p)^2 + b\frac{P_{out}I_p}{\tau V_s}} \right), \]

where \( a \) and \( b \) are given by (2.34). The network parameters in the testing conditions considered in this thesis are always such that the current slope and the voltage supply evaluated at \( t^* \) have the same sign. With reference to Figure 2.30, the current slope is negative \( (\dot{I}^* < 0) \) and so is voltage supply. Analogous conclusions, with all signs reverted, are drawn by considering the case \( \dot{I}^* > 0 \). This implies

\[ \text{sgn}(\dot{I}^*) = \text{sgn}(V_s \sin(-\sqrt{2}\omega \tau + \theta)) = \text{sgn}(a) \]

and allows choosing the correct root for \( \dot{I}^* \) in the form

\[ \dot{I}^* = \frac{\text{sgn}(a)}{2} \left( |a|I_p + \sqrt{(aI_p)^2 + b\frac{P_{out}I_p}{\tau V_s}} \right), \]

where the identity \( a \equiv \text{sgn}(a) \cdot |a| \) has been used, and whence (2.33) follows. \( \square \)

From (2.33) it follows that an increase of the prospective current \( I_p \) increases \( |\dot{I}^*| \), as can be expected since this makes the test more severe. On the other hand, in the spirit of this argumentation the cooling power \( P_{out} \) acts the opposite way, since a less good breaker with lower arc quenching properties (lower \( P_{out} \)), would be characterized, \textit{ceteris paribus}, by a lower \( |\dot{I}^*| \). This two counterbalancing effects are a typical of LV interruptions, since for large supply voltages the additive term containing \( P_{out} \) and \( I_p \) vanishes, as becomes evident from taking the limit of (2.33) for \( V_s \to \infty \). This accounts for saying that, unlike in HV (and usually also MV), in LV limiting breakers the arc current is definitely influenced by the presence of the arc.
In conclusion, the experimental data collected strongly suggests that the fate of interruption is decided in the relatively short period close to current zero and microscopic evaluators (with the exception of the time derivative of the current before current zero), which are built accordingly, do fulfill their task very encouragingly. The apparently so valuable information, hidden in the post-arc region, requires a very sensitive measuring device and a careful post-processing of its signal. We have adopted a high sensitivity current probe and Savitsky-Golay filtering, respectively, obtaining good and stable results.

2.5.3 Stability and Repeatability of Experiments

Stability and repeatability of low voltage tests have been explored by means of thirty tests divided in three groups of ten experiments. Each group is characterized by a different test severity, according to the probability of a thermal re-ignition immediately after the first current zero. In this way we accomplish two tasks: first, we study the statistical spread among a group of breakers which are nominally identical. Second, we deduce how this statistical spread varies with the test severity.

With “nominally identical” we refer to specimens of the same breaker whose inescapable differences fall below the tolerance ranges normally accepted for industrial production and quality check. This makes sense since we are interested in studying the repeatability of the interruption process from an industrial point of view, taking into account all the physiological differences that ordinary breakers may have. Thus, the loss of information about the stability of the arc behavior as an isolated process is counterbalanced by a major understanding of the arc behavior as part of a real industrial process. The circuit breaker under test is ABB T3.

Table 2.4 summarizes the experiments that have been carried out. The breaker has three poles and single pole tests have been carried out (see §2.2). Each specimen has been tested three times, one per pole. Voltage and current have been varied proportionally in order to keep the network impedance constant. Thus, all the left poles were tested at 6.2kA@340V, the central poles at 6.9kA@380V and the right poles at 9.1kA@500V. The power factor has been set to 0.52 for all the tests. These values were chosen to explore the behavior of the breakers in three different regions of increasing failure probability. In particular, the central poles were meant to be tested in a borderline situation, even if the results show that the test severity could have been increased a little bit to see some sporadic (thermal) failure.

In order to study the repeatability of our experiments we first have to choose some quantitative parameter to conduct a standard statistical analysis.
### Table 2.4: Overview of the experimental campaign carried out to assess the stability and repeatability of arc tests (L/C/R = left/central/right pole; OK/KO = successful/failed interruption at the first current zero).

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Pole</th>
<th>Test Voltage [V]</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>L</td>
<td>340</td>
<td>OK</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>380</td>
<td>OK</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>500</td>
<td>KO</td>
</tr>
<tr>
<td>2</td>
<td>L</td>
<td>340</td>
<td>OK</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>380</td>
<td>OK</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>500</td>
<td>KO</td>
</tr>
<tr>
<td>3</td>
<td>L</td>
<td>340</td>
<td>OK</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>380</td>
<td>OK</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>500</td>
<td>KO</td>
</tr>
<tr>
<td>4</td>
<td>L</td>
<td>340</td>
<td>OK</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>380</td>
<td>OK</td>
</tr>
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<td></td>
<td>R</td>
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<td>5</td>
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<td>6</td>
<td>L</td>
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<td>7</td>
<td>L</td>
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<td></td>
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<td>8</td>
<td>L</td>
<td>340</td>
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<tr>
<td></td>
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<tr>
<td>9</td>
<td>L</td>
<td>340</td>
<td>OK</td>
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<tr>
<td></td>
<td>C</td>
<td>380</td>
<td>OK</td>
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<tr>
<td></td>
<td>R</td>
<td>500</td>
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<td>10</td>
<td>L</td>
<td>340</td>
<td>OK</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>380</td>
<td>OK</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>500</td>
<td>KO</td>
</tr>
</tbody>
</table>

From this point of view, the most intuitive way to characterize a short-circuit test is to consider the outcome of the interruption, which always gives a clear OK/KO response. This criterion is, indeed, very rough, as it fails whenever the experiments are conducted far from the limit of the breaker. Thus, two specimens with very different interruption performances may be judged to be similar only because the testing conditions are not severe enough to make one of them fail. For this reason, we chose to focus on the first four microscopic indicators defined in §2.5, which seem to be quite related to the interruption quality of each test. The other indicators will not be considered anymore, as they appear to be useless to summarize the extinction phenomena after the current zero. Figures from 2.32 to 2.35 show the results of measurements for each indicator, divided in three groups of tests according to Table 2.4. Finally,
Table 2.5 summarizes the results in a compact and easy-to-read way.

The first comment to be done is about the ability of our microscopic evaluators, defined in §2.5.2, to separate successful interruption from failures. This is a basic requirement that every evaluator should have, i.e., to show a sufficiently clear threshold with respect to the outcome of the interruption. In §2.5.2 we defined four microscopic indicators (the current slope at current zero, the electric charge through the post-arc channel, the corresponding Joule’s integral and the arc conductance at current zero) which were very satisfactory from this point of view. With this new experiments, our previous conclusions about success and failure detection are very well confirmed for the first three indicators, while for the arc conductance the failure region tends to overlap slightly (but not dramatically) with the successful interruption one.

Nonetheless, all the indicators show that test n.3, corresponding to the lowest black square among the experiments done at 500 V, although it is a failure, lies completely in the region of successful interruptions. Figure 2.36 shows three current signals relevant to three 6.2kA@340 V tests. The dash-dotted line is relevant to a successful interruption and the corresponding indicators fall in the successful interruption region (white squares in Figures from 2.32 to 2.35). The dashed line is relevant to a failure which occurs immediately after the current zero and it is thus reasonable to infer that the causes of the missed interruption be thermal, namely the temperature of the conducting path is still too high to lead to a complete arc extinction. The proposed indicators are capable to detect such kind of divergence and, as a consequence, their values fall in the missed interruption region (black squares in Figures from 2.32 to 2.35). The solid line is relevant to the above mentioned test n.3. Initially, the behavior is very similar to the successful interruption case, with the two relevant current signals practically in complete superposition. This suggests that, as far thermal issues are concerned, the interruption should have been successful and justifies the low value found for the indicators, falling in the successful interruption region (white squares in Figures from 2.32 to 2.35). Suddenly the interruption fails (determining the black color of the relevant square in Figures from 2.32 to 2.35).

<table>
<thead>
<tr>
<th>Performance Evaluator</th>
<th>340V</th>
<th>380V</th>
<th>500V</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_0$ [A/s]</td>
<td>(0.82±0.29) · 10^6</td>
<td>(1.18±0.21) · 10^6</td>
<td>(1.60±0.31) · 10^6</td>
</tr>
<tr>
<td>$\eta_7$ [C]</td>
<td>(1.45±0.53) · 10^2</td>
<td>(2.72±0.73) · 10^2</td>
<td>(5.61±1.48) · 10^2</td>
</tr>
<tr>
<td>$\eta_8$ [A^2 s]</td>
<td>(0.22±0.13) · 10^{-3}</td>
<td>(0.72±0.30) · 10^{-3}</td>
<td>(3.20±1.63) · 10^{-3}</td>
</tr>
<tr>
<td>$\eta_9$ [S]</td>
<td>(1.71±0.80) · 10^{-2}</td>
<td>(3.05±1.01) · 10^{-2}</td>
<td>(4.55±1.87) · 10^{-2}</td>
</tr>
</tbody>
</table>

Table 2.5: Average and standard deviation of microscopic performance evaluators at low, medium and high testing condition severity.
Figure 2.32: Current slope at CZ [A/s] at low, medium and high testing condition severity (white squares denote successes, black denote failures), with average and standard deviation.

Figure 2.33: Electric charge through the post-arc channel [C] at low, medium and high testing condition severity (white squares denote successes, black denote failures), with average and standard deviation.
Figure 2.34: Joule integral \( [A^2s] \) at low, medium and high testing condition severity (white squares denote successes, black denote failures), with average and standard deviation.

Figure 2.35: Conductance at CZ [S] at low, medium and high testing condition severity (white squares denote successes, black denote failures), with average and standard deviation.
Figure 2.36: Current signal for three 6.2kA@340V tests: a successful interruption (dash-dotted), a failure due to thermal restrike (dashed) and a failure probably due to non thermal phenomena (solid).

2.35) for something which happens approximately 100µs after the current zero, and thus which is undetectable by our indicators. This is because the latter are computed either punctually at the current zero or based on the first 10µs thereafter.

The observed behavior for test n.3 suggests to relate the causes of the missed interruption to issues other than thermal ones. It is not possible from the sole examination of the oscillograms to understand what actually went wrong during the interruption and the list of possible causes is long and difficult to be verified. We may conclude that a series of other phenomena exist that complicate and spoil the whole picture. From the actually very restricted number of tests carried out, one could infer that the presence of such disturbances be relatively limited to a few percent. A more thorough experimental analysis would be required to draw more definitive conclusions.

The results also show a remarkable statistical spread, quantified by the stan-
2.5. PERFORMANCE EVALUATORS

Figure 2.37: Comparison between three nominally identical 9.1 kA@500 V tests.

dard deviation of each microscopic indicator (see Table 2.5) which is greater than half the mean value in more than one case. We now see a crucial point in our analysis, that is, the arc interruption, seen as an industrial phenomenon, is widely stochastic. This may be appreciated, for instance, from Figure 2.37, where three out the ten 9.1 kA@500 V tests are compared. The current zero of the three oscillograms have been synchronized and a non negligible difference may be observed both in voltage and in current signals, despite both the breakers and the testing conditions are nominally identical. Of course we do not claim that it is impossible to achieve a satisfactory repeatability and stability of the interruption process. Probably, it is possible to build a very simple circuit breaker, with a very simple moving mechanism and a very simple arc chamber, so that the sources of instability are reduced as much as possible.
But this would not be the spirit of this analysis, which, instead, is to find a way to measure quantitatively the interruption performances of a restricted set of industrial specimens and gather high-quality information to reduce the number of future tests. So we have necessarily to deal with real-life devices, coming from real factories and subjected to all the tolerances accepted by the manufactures. From this point of view, the instability of low voltage arc interruptions may be considered to be *intrinsic*, and every arc model will inherit this “original sin” (see §5.8).

It is of interest to investigate the possible causes that make the difference in between nominally identical circuit breakers. Nonetheless, this kind of analysis would require a massive experimental champaign, along with fine inspection of the components of the tested circuit breakers, which would fall out of the scope of the present thesis. Suspicion is aroused by some technologically critical features which are accomplished by technologically rough components, such as the magnetic pulling effect performed by the rack of splitter plates onto the electric arc (see, e.g., §1.1.3). It is known that the ferromagnetic material of the splitter plates induces a magnetic perturbation that helps bringing the arc inside the extinguishing chamber, thus playing a crucial role in the successf-ness of the current interruption. On the other hand, splitter plates are made of common iron, with poor or no quality check required from the suppliers. Furthermore, suppliers can also be different, and with different industrial production processes. A similar argument also holds for other components of the circuit breaker. Also, it is very difficult to guarantee that the testing conditions be truly similar, under the precision required for our study.

As mentioned before, this test campaign aimed also at investigating the repeatability of experiments as a function of the test severity. Form Table 2.5 we see that the standard deviation increases with the voltage supply. However, the coefficient of variation, i.e., the ratio of the standard deviation and the mean value, is fairly constant (or even decreasing with voltage). We conclude this section by noting that the information gathered with these measurements is not sufficient to decide which indicator is more appropriate to judge the arc extinction. It is obvious that the current slope and the arc conductance at current zero require a careful data post-processing, such as filtering and offset removing. On the other hand, the electric charge through the post-arc channel and the corresponding Joule’s integral are evaluated in a more straightforward way.
2.6 Conclusions

The current chapter reports the experimental part of our research, mostly consisting of electrical measurements, possibly supplemented by optical observations by means of fiber optics.

A lumped parameter description has been produced to model the test network, resulting in a RLC circuit. The model is satisfactory for current transients on time scales of the order of milliseconds, or above. On the other hand, the model proved to be not adequate to describe very fast transients, of the order of tenths of microseconds, which are typical of current interruptions. Particularly, the amplitude of the transient recovery voltage is poorly resolved. Owing to the importance of such an issue when applying arc models to circuit breakers, a finer model should be developed.

A fiber optics inspection technique has been developed and applied to arc imaging inside circuit breakers. Only tiny holes have to be drilled on circuit breaker sidewalls, to host the fibers. For this reason the method is found attractive because it is less intrusive than fast camera imaging. An original method has also been developed to reconstruct a smooth image of the arc plasma cloud, starting from a discrete set of observation points.

Several performance indicators have been defined and tested, in order to assess the interruption quality. It turns out that only those indicators based on data measured during the current zero period are experimentally correlated with the successfulness of the interruption. The experiments also show a very stochastic behavior of the arc, even in nominally identical testing conditions.
Chapter 3

Low Voltage Arc Physics

3.1 Summary

A short account of the basic physics of the electric arc in low voltage circuit breakers is outlined in this chapter, yielding the picture of a thermal plasma in high pressure conditions (that is, close to atmospheric pressure conditions). First, the chemical composition of the air plasma mixture is described, defining the interacting species. Then, a Lagrangian description at the particle level is sketched, so to understand the mechanism of energy transfer through collisions, especially in between particle species of similar mass, and the mechanism of energy gain from the electric field, especially for particle species of lighter mass, i.e., electrons.

The Eulerian plasma kinetic approach is introduced, leading to Boltzmann equation in a collisional plasma. The moments of Boltzmann equation are used to deduce general transport equations governing the conservation of mass, momentum and energy, per each species. The phenomenological closure of the system by means of the equation of state is discussed. Maxwell equations are required to describe the electromagnetic portion of the problem, which enters the momentum conservation equation in the form of Lorentz force and the energy conservation equation if the form of Joule heating. The ionization-recombination equilibrium in the plasma phase is described by means of Saha equation, from which a temperature (and pressure) dependent electric conductivity ultimately results.
CHAPTER 3. LOW VOLTAGE ARC PHYSICS

The description is thus simplified from the multi-fluid level to a two-fluid model, accounting for electrons and heavy particles (positive ions and neutrals). This level is fine enough to understand both Debye shielding of electric charges and how the balance between energy gain and energy transfer influences the plasma behavior in non equilibrium conditions. The stemming theory, termed non equilibrium theory, is necessary to describe the behavior of the arc when close to extinction and at its roots, that is, the arc-electrode interfaces.

In the rest of the plasma column and in conditions different from close to the current zero, the single fluid description can be adopted, which is based on the local thermal equilibrium (LTE) hypothesis. The resulting magnetohydrodynamic (MHD) model is computationally tractable on modern computers, in order to describe the macroscopic scale of the arc dynamics as a conducting, compressible, viscid fluid, driven by electromagnetic forces and pressure gradients. Navier-Stokes equations, governing the flow, and Maxwell equations, governing electromagnetism, can be handled by means of a weakly coupled approach, since the magnetic Reynolds number can be shown to be low in the typical conditions which are expected in low voltage circuit breakers. Due to the very high temperatures reached inside the arc plasma, approximately in the $10^4 - 20^4 K$ range, radiation is the dominant heat transfer mechanism. As a consequence, a suitably accurate model is needed to adequately resolve the temperature field, on which, in turns, the arc conductance is depending. The problem of radiative heat transfer in a participating medium is introduced and addressed by means of a photohydrodynamic approach ($P_1$ model), which is basically a truncated spherical harmonic expansion approximation of the exact equations.

The qualitative description of arc root physics is attempted in order to provide a base for a conceptual model to account for the macro scale effects, without resolving the micro scale physics inside a macro scale approach like MHD. Arc roots are responsible for a strong resistance increment when the arc enters a region of the breaker in which it is broken into many serial branches. The importance of a correct modeling of this issue is well understood, since this is the core technology adopted by low voltage circuit breaker manufacturers to produce current limiting devices and quench the arc. Also, a brief overview is provided for material ablation, which is due to the strong, arc-originated heating of plastic and metallic parts inside the circuit breaker. The topic is of interest since nowadays gassing materials are largely and purposely added in low voltage circuit breakers, in order to increase their performances.
3.2 An Introduction to Air Plasma

Contrarily to the high- and medium-voltage world, where the current interruption takes place either in vacuum or in a suitably engineered gas atmosphere (typically SF$_6$, or CO$_2$ in a possible future to come), low voltage circuit breakers are not insulated devices and the interruption takes place in a normal air atmosphere. This can be easily understood since voltage is - indeed - low in low voltage, and LV circuit breakers do not have to withstand the very strong electric fields experienced in high- and medium-voltage circuit breakers. As a consequence, there is no economical drive pushing circuit breaker manufacturers to adopt different dielectric media than the common surrounding air environment. In order words, the size of low voltage circuit breakers is bounded from below by other constraints than the dielectric rigidity of the insulating medium.

3.2.1 The Chemistry of Low Voltage Air Plasma

Air is a gas mixture essentially composed of nitrogen (N$_2$) and oxygen (O$_2$), in the ratio 4:1. More precisely, the average composition of dry air atmosphere, by volume, in standard conditions for temperature and pressure (273.15 K and 100 kPa, respectively [94]) is reported in Table 3.1 [140]. On earth surface, water vapor content ranges in between 1% and 3%, by volume, depending on the place and the season [140]. From the above picture and with reference to air plasma modeling, all other species may be neglected but nitrogen, oxygen and the products of their dissociation, ionization and recombination at higher temperatures, as hereafter illustrated.

When sufficient energy is provided, i.e., at the high temperatures (approximately > 2000 K) produced by Joule heating in the case of circuit breakers, chemical reactions take place and N$_2$ and O$_2$ molecules first dissociate into electrically neutral atoms, namely N and O. Recombinations also occur in between nitrogen and oxygen atoms to form electrically neutral nitrogen (mon)oxide NO molecules. At higher temperatures (approximately > 9000 K), electrically neutral molecules are ionized into positive ions N$^+$ and O$^+$, and negatively charged electrons e$^-$ are released. An air plasma is formed. The presence of charge carriers, i.e., positive ions and, above all, free electrons, makes the air plasma electrically conductive. At even higher temperatures (approximately > 20000 K), nitrogen ions start being subject to the second ionization and N$^{++}$ ions are formed, together with the release of additional electrons.

The dissociation, recombination and ionization processes of air plasma are shown in Figure 3.1 [124]. Particle numbers are here drawn as a function of
temperature and relatively to standard, undissociated conditions. The total number of particle, $n$, is a monotonically non decreasing function (left scale). All other particle species are referred to the right scale, expressing their relative contribution, always with reference to the undissociated state. In the low temperature range ($\leq 2000K$), virtually only $N_2$ and $O_2$ are present, in the 4:1 ratio. At progressively increasing temperatures, the first species to dissociate is oxygen, according to the reaction

$$O_2 \rightarrow 2O \tag{3.1}$$

and starting from in between $2000K$ and $3000K$, followed by nitrogen, according to the reaction

$$N_2 \rightarrow 2N \tag{3.2}$$

and starting from in between $4000K$ and $5000K$. Stoichiometric coefficient 2 explains the doubling in particle numbers when the atomic forms of both the two elements are produced. Oxygen molecule $O_2$ (or dioxide) has a bond energy equal to $498.3kJ/mol$, whilst nitrogen molecule $N_2$ has a higher bond energy, equal to $945kJ/mol$, which is a very high value yielding strong stability to the molecule [140]. This explains why the latter species is later dissociated compared with the former.

<table>
<thead>
<tr>
<th>Species</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nitrogen ($N_2$)</td>
<td>780,840.00 ppmv (78.084000%)</td>
</tr>
<tr>
<td>Oxygen ($O_2$)</td>
<td>209,460.00 ppmv (20.946000%)</td>
</tr>
<tr>
<td>Argon ($Ar$)</td>
<td>9,340.00 ppmv (0.934000%)</td>
</tr>
<tr>
<td>Carbon dioxide ($CO_2$)</td>
<td>383.00 ppmv (0.038300%)</td>
</tr>
<tr>
<td>Neon ($Ne$)</td>
<td>18.18 ppmv (0.001818%)</td>
</tr>
<tr>
<td>Helium ($He$)</td>
<td>5.24 ppmv (0.000524%)</td>
</tr>
<tr>
<td>Methane ($CH_4$)</td>
<td>1.75 ppmv (0.000175%)</td>
</tr>
<tr>
<td>Krypton ($Kr$)</td>
<td>1.14 ppmv (0.000114%)</td>
</tr>
<tr>
<td>Hydrogen ($H_2$)</td>
<td>0.55 ppmv (0.000055%)</td>
</tr>
<tr>
<td>Dinitrogen oxide ($N_2O$)</td>
<td>0.30 ppmv (0.000030%)</td>
</tr>
<tr>
<td>Xenon ($Xe$)</td>
<td>0.09 ppmv (0.00009%)</td>
</tr>
<tr>
<td>Ozone ($O_3$)</td>
<td>$\leq 0.07$ ppmv ($\leq 0.00007%$)</td>
</tr>
<tr>
<td>Nitrogen dioxide ($NO_2$)</td>
<td>0.02 ppmv (0.000002%)</td>
</tr>
<tr>
<td>Iodine ($I$)</td>
<td>0.01 ppmv (0.000001%)</td>
</tr>
<tr>
<td>Carbon monoxide ($CO$)</td>
<td>trace</td>
</tr>
<tr>
<td>Ammonia ($NH_3$)</td>
<td>trace</td>
</tr>
</tbody>
</table>

Table 3.1: Average composition of dry air atmosphere, by volume, in standard conditions for temperature and pressure, where ppmv means parts per million by volume [140].
In the temperature range from 2000\textit{K} and 6000\textit{K}, a weak bump in nitrogen monoxide (NO) concentration is observed. Nitrogen monoxide is produced from molecular nitrogen and oxygen according to the reaction

\[ \text{N}_2 + \text{O}_2 \rightleftharpoons 2\text{NO}, \quad (3.3) \]
which in standard conditions has a Gibbs free energy of formation equal to $\Delta G_f^0 = 86.6 \text{kJ/mol} > 0$ [140]. The positive value means that the reaction is thermodynamically unstable, and thus not spontaneous. At 3000K the equilibrium of reaction (3.3) is shifted to approximately 14% NO [140]. This explains the limited presence of NO in the gas mixture. For higher temperatures nitrogen and oxygen find a more stable configuration than bonding into NO and this explains the progressive disappearance of nitrogen monoxide. Until the beginning of the last century, the electric arc has been industrially exploited to induce reaction (3.3), according to the very pollutant Birkeland-Eyde process.

One could be surprised not to observe a significant presence in the gas mixture of nitrogen dioxide NO$_2$, which, together with nitrogen oxide NO, is produced in internal combustion engines (the two species are partially removed by catalytic converters). Actually, in standard conditions, despite NO$_2$ being unstable, with Gibbs free energy of formation equal to $\Delta G_f^0 = 51.3 \text{kJ/mol} > 0$, the reaction

$$2\text{NO} + \text{O}_2 \rightleftharpoons 2\text{NO}_2$$

(3.4)

indeed has free Gibbs energy of production equal to $\Delta G^0 = -69.9 \text{kJ/mol} < 0$, which implies a thermodynamically spontaneous process [140]. Nevertheless, at temperatures higher than 600°C, and thus a fortiori at characteristic arc temperatures, the chemical balance is almost completely shifted to the left in reaction (3.4) [140]. As already seen, it is only for higher temperatures that the - yet low - concentration of NO is non negligible and, meanwhile, O$_2$ concentration starts to disappear. This explains the extreme scarcity of nitrogen dioxide.

At temperatures around 7000K both nitrogen and oxygen are nearly fully dissociated into single atoms. Neutrality is maintained for some other 2000K during which the relevant atomic particle number curves remain nearly constant at twice as high a value as the initial value of the molecular particle number curve. The first three ionization energies of nitrogen and oxygen atoms are reported in Table 3.2 [140]. The first ionization energy is similar in the two cases, which explains why both nitrogen and oxygen starts being ionized approximately at the same temperature, around 9000K, and according to the reactions

$$\text{N} \rightarrow \text{N}^+ + e^-$$

(3.5)

and

$$\text{O} \rightarrow \text{O}^+ + e^-.$$  

(3.6)

The atomic nitrogen particle number curve is observed to decline faster than the atomic oxygen one, due to the 4:1 ratio in the concentration of the two species.
The second ionization requires a considerably higher energy in both cases, and particularly for oxygen. As a consequence, only a minor portion of ionic nitrogen gets further ionized according to the reaction

\[ \text{N}^+ \rightarrow \text{N}^{++} + e^-, \]  

at temperatures higher than 20000 K, where a slight negative bend of \( \text{N}^+ \) particle number curve is observed. The second ionization of oxygen and the subsequent ionizations of both oxygen and nitrogen require too high energies to be significantly occurring at typical arc temperatures. Of course, more energetic particles can be further ionized, but their concentration is negligible. It follows that most of the positive ions in a low voltage air plasma are \( \text{N}^+ \) ions, with the electric charge of one electron (and opposite sign, obviously). It is apparent that full ionization is never present in a low voltage arc plasma.

The chemistry so far outlined is relevant to a pure air plasma. Low voltage circuit breaker manufacturers purposely introduce gassing material to increase arc quenching properties and electronegativity. Also the metal parts and plastic enclosures get ablated by the heat produced by the arc. As a consequence, much more different species are introduced into the gas mixture so that the real chemistry inside a circuit breaker is way more complicated. Probably, the situation is not fully dominated at the technical level, where blind trial and error is performed, and further, advanced studies are required.

### 3.3 The Particle Description

A physically very simple description of plasma, though computationally impracticable, is obtained at the particle scale. Many kind of chemical species have to be accounted for, namely (according to §3.2.1) electrons \( e^- \), each type of ions, viz. \( \text{N}^+ \), \( \text{N}^{++} \), \( \text{O}^+ \), and each type of neutrals, viz. \( \text{N}_2 \), \( \text{O}_2 \), NO. Any other species could be included into the count, if necessary, like, e.g., ablated material gasses from plastic and metallic parts. Formally, we define the set \( S \) to include all of the types of particle species.
3.3.1 Models of Particles

A first, natural question arise as to what level of detail is needed when describing a “particle”. As usual, an “exact” description would require much more complexity than what we can afford to solve an engineering problem, so that one must choose based on what is pursuing. Since our goal is simply to show how to deduce the equations to solve and obtain some results that may yield some insight into the complex behavior of the arc plasma, at this stage we will limit ourself to the most simple and straightforward representation of a point shaped particle, and later show, when dealing with a computational approach, how to correct the problem so to retain the physics we need. According to this classical (meaning, not quantum mechanical) point of view, any particle is fully described by its position vector $\mathbf{x}$ and its velocity vector $\mathbf{v}$. This model is well suited for a mono-atomic particle gas at sufficiently low temperature. From the Hamiltonian mechanics standpoint, the relevant phase space for a single particle of this kind is $(\mathbf{x}, \mathbf{v})$ and thus has dimension 6. An ensemble of $n$ particles would lie in a phase space of dimension $6n$.

Even though we will stick to such a simplistic model when writing the equations, it is important to understand that it is inadequate for the bi-atomic particle gasses, such as $\text{N}_2$, $\text{O}_2$ and NO, that constitute the fractions of low voltage air plasma. Two additional degrees of freedom are required to describe the orientation of the molecule (e.g., two angles to define the molecule axis), and other two to describe its angular velocity of rotation (e.g., the derivatives of the two angles). The phase space should thus be of dimension 10 for the single particle and $10n$ for the whole gas.

From §3.2.1, we know that the presence of more complex, but still rigid molecules can be neglected, at least as long as gassing materials are not accounted for. Nonetheless, we remind that the general, fully 3-dimensional molecule would have 12 degrees of freedom (e.g., 3 centroid positions, 3 orientation angles and the relevant derivatives) and the whole gas would have $12n$.

Actually, also the rigid body conception for a molecule is strictly speaking inadequate at the high plasma temperatures, since the distance in between atoms is oscillating. In the case of bi-atomic gasses, two additional degrees of freedom are required, such as the atomic distance and its derivative. In the simplistic assumption to treat atomic oscillations as elastic, an harmonic oscillator is obtained with energy being cyclically swapped back and forth from kinetic to potential and vice versa.

By definition, a plasma exists at temperatures such that some electrons are removed by their shells. Thus, intrinsically, the atoms themselves are not
rigorously conceivable as point particles, for more than a simple perturbation is produced over their electronic structure. This is also the limit of classical models, since quantum mechanics is required from this point on, revealing other degrees of freedom. Generally speaking, as temperature is increased, more and more degrees of freedom are activated, corresponding to physical motions described by the physics of increasingly finer structures of matter. Going the other way around, at the absolute zero all of the degrees of freedom would be frozen, with matter deprived of any possibility of motion.

Lengthening the list of degrees of freedom not only makes the governing equation bigger and more difficult. For Boltzmann equipartition theorem, the energy received from the interaction with surrounding particles and environment is, on average, evenly distributed in between all available possibilities of motion. Consequently, the specific heat of an electric arc made of complex molecule gas plasma is several times higher than in the mono-atomic particle case and we will correct the stemming energy balance fluid equations so to account for it. This is an example of how the physical micro scale may largely affect the engineering macro scale.

3.3.2 Charged Particles Moving in Electromagnetic Fields

When examined at the particle scale, plasma is essentially a set of charged particles interacting by means of electromagnetic forces. In principle, one has to write a system of equations of motion for the single particle. Thus, assuming that particles may be assimilated to points, let \( j, s \) be the indexes of the \( j \)-th particle of the species \( s \), and \( m_s, q_s \) be, respectively, the mass and the charge of the \( s \)-th species. The equations governing the motion of the \( j \)-th particle are then

\[
\begin{align*}
\dot{x}_{j,s}(t) &= v_{j,s}(t) \\
m_s \dot{v}_{j,s}(t) &= q_s \left( E(x_{j,s}(t), t) + v_{j,s}(t) \times B(x_{j,s}(t), t) \right),
\end{align*}
\]

where \( x_{j,s}(t) \) and \( v_{j,s}(t) \) are, respectively, the position and velocity of particle \( j, s \) at time \( t \).

The r.h.s. of the second equation (Newton’s law of dynamics) of (3.8) is given by Lorentz force, where \( E(x_{j,s}(t), t) \) and \( B(x_{j,s}(t), t) \) are, respectively, the electric field and magnetic flux density perceived by the single particle when in point \( x_{j,s} \) at time \( t \). Contrarily to what experienced at the macro-scale, where electric and magnetic fields are smooth, the micro-scale picture is definitely irregular. If we fix a point \( x \) in space, the electromagnetic field at that point is the superposition of a generally smooth contribution due to external charges and currents, a generally smooth contribution due to the multitude of plasma
particles far away from $x$ (the so-called far field), and finally an extremely spiky, as well as fast and chaotically varying contribution due to the particles close to $x$. The latter contribution is spiky because the well-known solution for the electric field generated from a stationary point charge and the solution for the magnetic field generated from a moving point charge both contain a mathematical singularity of type $1/r^2$ with reference to the distance $r$ from the charge position. It is chaotically and fast varying because of collisions in between particles.

A comparative analysis of different particle data is important to understand the behavior of plasma. The rest mass of electrons is $m_e = 9.109 \cdot 10^{-31} \text{kg}$ [100, 140], while that of protons is $m_p = 1.672 \cdot 10^{-27} \text{kg}$ [140]. Neutrons have a rest mass similar to protons, being $m_n = 1.675 \cdot 10^{-27} \text{kg}$ [140]. Therefore the ratio between the mass of a proton, or a neutron, and that of an electron is about 1836 (often approximated to 2000). The unbalance is even much greater when the neutrals and ions occurring in air plasma are accounted for. Nitrogen has atomic number 7 and average (over different isotopes) neutron number approximately equal to 7, so that the average mass number is approximately 14 [140]. Oxygen has atomic number 8 and average (over different isotopes) neutron number approximately equal to 8, so that the average mass number is approximately 16 [140]. Based on the average mass numbers of elements, a comparison between the molecular mass of the main chemical species present in air plasma is reported in Table 3.3, where the ratio is also shown with reference to the mass of the electron. A frequent simplifying approximation in plasma physics is that of distinguishing between electrons and the class of heavy particles, namely neutrals and (positive) ions, hereafter denoted by subscript $h$. When the general classes of ions and neutrals are concerned, subscripts $i$ and $n$ will be used, respectively. From the above evaluations, heavy particles (and particularly ions) are at least 4 orders of magnitude heavier than electrons, and thus

$$m_e \ll m_h, \quad m_e \ll m_i, \quad m_e \ll m_n.$$  \hfill (3.9)

In the followings, the approximations

$$\frac{m_e}{m_h} \approx 0, \quad \frac{m_e}{m_i} \approx 0, \quad \frac{m_e}{m_n} \approx 0.$$  \hfill (3.10)

will be used extensively.

Such great differences in inertial properties are not counterbalanced by the electric charge. To handle the general case, if the elementary charge $e^- = 1.60218 \cdot 10^{-19} \text{C}$ [100, 140] is introduced\(^1\), the particle charge may be expressed as

$$q_s := Z_se^-,$$  \hfill (3.11)

\(^1\)the “$e$” superscript is here introduced to avoid misunderstanding with Neper’s number.
3.3. **THE PARTICLE DESCRIPTION**

<table>
<thead>
<tr>
<th>Species</th>
<th>Molar Mass</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrons</td>
<td>(e⁻)</td>
<td>(1/1836)u</td>
</tr>
<tr>
<td>Atomic nitrogen and its ions</td>
<td>(N, N⁺, N⁺⁺)</td>
<td>14u</td>
</tr>
<tr>
<td>Atomic oxygen and its ions</td>
<td>(O, O⁺)</td>
<td>16u</td>
</tr>
<tr>
<td>Molecular nitrogen</td>
<td>(N₂)</td>
<td>28u</td>
</tr>
<tr>
<td>Nitrogen monoxide</td>
<td>(NO)</td>
<td>30u</td>
</tr>
<tr>
<td>Molecular oxygen</td>
<td>(O₂)</td>
<td>32u</td>
</tr>
</tbody>
</table>

Table 3.3: Comparison of the molecular mass (approximate value, in unified atomic mass unit u) of the main chemical species in air plasma, and their ratio with reference to that of electrons.

where \( Z_s \) is the *charge number* of the particle. From what above, \( Z_e = -1 \), while \( Z_s = 1 \) for \( N^+ \) and \( O^+ \), \( Z_s = 2 \) for \( N^{++} \) and, finally, \( Z_s = 0 \) for all neutrals. In §3.2.1 we have outlined that the vast majority of positive ions has only one electron removed, with the exceptions of the few \( N^{++} \) ions, so that in the case of the *average charge number* \( Z_i \) for (positive) ions the approximation

\[
Z_i \approx 1
\]  

(3.12)

holds and, on a statistical base, the electric charge of ions and electrons are opposite in sign and equal in module. Consequently,

\[
|q_e| \approx |q_i|.
\]

When considering the class of heavy particles in general, averaging ions with neutrals, one may rely on

\[
Z_h \in [0, 1],
\]  

(3.13)

with \( Z_h = 0 \) when the gas is not ionized at all and \( Z_h = 1 \) when all neutrals have completely disappeared.

Apart from neutral particles, which are unaffected by Lorentz force, the picture is thus that of particles with very different mass subject to the same or similar Lorentz force. The magnetic part of Lorentz force is orthogonal to the particle velocity and thus results in null work, with no increase in the particle velocity. The electric field accelerates charged particles and from (3.8) it follows that electrons can reach much higher velocities than heavy ions. Collisions in between the different species thermalize them, provided that the collision frequency is high enough with reference to the mean free path.

We have neglected gravitational interactions between particles, owing to their extreme weakness with reference to electromagnetic interactions. One can understand this by comparing the gravitational force \( F_G = Gm_1m_2/r^2 \) with, e.g., Coulomb force \( F_C = q_1q_2/(4\pi\varepsilon_0r^2) \). Considering particles having masses
of order $m_1 = m_2 \approx 10^{-26}$, i.e., the heaviest ions in air plasma, and charge of order $q_1 = q_2 \approx e^-$, one gets a gravitational force of order $F_G \approx 10^{-62}/r^2$ and a Coulomb force of order $F_C \approx 10^{-28}/r^2$, in SI units. On the other hand, the effect of the external gravitational field (as well as some other external constant field) on the arc plasma may be easily accounted for by an additional contribution into the equation of motion (3.8).

### 3.3.3 Particle Energy Transfer Through Collisions

Arc plasma particles may have very different velocity and kinetic energy, but collisions tend to transfer energy from faster to slower particles. This energy redistribution mechanism is not quantitatively equal for any two interacting particles, for ceteris paribus those with a similar mass exchange a bigger amount of kinetic energy than those with an unbalanced mass ratio.

Particle collision are actually distance interactions and trajectory deviation ruled by Coulomb force. Such kind of interactions are termed Coulomb collisions. The problem is known as Kepler’s problem and is a special case of the two-body problem in which the two bodies interact with a central force proportional to the square of their inverse distance. The exact solution [30] would also require quantum mechanics to describe close particle interaction. To understand the energy redistribution mechanism one can actually consider
3.3. THE PARTICLE DESCRIPTION

a much simpler exposition, also because the two-body problem is itself an abstraction, for in the real plasma gas of particles the interaction is continuous and perpetual and involves a multitude of bodies at the same time. Moreover, we are intentionally leaving out of consideration those collisions which are energetic enough to release an electron and produce a ionization.

Therefore we refer to the situation schematically shown in Figure 3.2, where a particle with mass \( m \) and electric charge \( q \) travels with speed \( v_0 \) toward a particle with mass \( M \) and electric charge \( Q \), initially at rest. If there were no interaction, then the undeflected trajectory of the first particle would have a distance of closest approach \( b \), termed impact parameter. We start considering the mass unbalanced case and assume that \( M \) is bigger enough than \( m \), so that we neglect the displacement of the bigger particle when defining the trajectory of the smaller one. From the exact problem solution, the smaller particle deviates its trajectory along a hyperbola and asymptotically reaches a direction tilted by an angle \( \vartheta \) from the original direction. The entity of such a deviation \( \vartheta \) may be computed with reference to the Coulomb force and thus depends on the impact parameter \( b \) and the two electric charges \( q \) and \( Q \). Obviously, a given particle undertakes a series of collisions with many others, according to each time different impact parameters. In Figure 3.2 the continuous trajectory holds when the two charges are opposite in sign, like an electron-ion collision, while the dotted trajectory is representative of a collision between particles with charges of the same sign.

After the collision, the small particle will have a velocity of magnitude \( v := |v| \), generally different from \( v_0 := |v_0| \), and the difference in momentum and kinetic energy will have been transferred to the bigger particle, resulting in a velocity \( V \) of the latter, described by magnitude \( V \) and anomaly \( \Theta \) (see Figure 3.2, where continuous and dotted arrow are relevant to the continuous and dotted deviated trajectory, respectively). For any possible value of \( \vartheta \), one can compute the final velocities (which is what we are interested in) by the principles of momentum and energy conservation, obtaining the algebraic system

\[
\begin{align*}
mv \cos \vartheta + MV \cos \Theta &= mv_0 \\
mv \sin \vartheta + MV \sin \Theta &= 0 \\
\frac{1}{2}mv^2 + \frac{1}{2}MV^2 &= \frac{1}{2}mv_0^2.
\end{align*}
\]

The problem is solved elementarily, and the final velocity of the smaller particle is found to be

\[
v = \alpha \left( \frac{m}{M}; \vartheta \right) \cdot v_0,
\]

where

\[
\alpha \left( \frac{m}{M}; \vartheta \right) := \frac{m}{M} \cos \vartheta + \frac{\sqrt{1 - \left( \frac{m}{M} \right)^2 \sin^2 \vartheta}}{1 + \frac{m}{M}}
\]
Figure 3.3: Energy transfer as a function of colliding particle mass ratio and for various values of impact parameter (through the deflection angle).

is the reduction ratio. The reduction in the kinetic energy of the smaller particle is given by

\[ K = \alpha^2 \left( \frac{m}{M} ; \vartheta \right) \cdot K_0, \]

where \( K_0 \) and \( K \) are kinetic energy before and after the collision, respectively.

We introduce the kinetic energy exchange coefficient

\[ \Xi \left( \frac{m}{M} ; \vartheta \right) := \frac{\Delta K}{K_0} = 1 - \alpha^2 \left( \frac{m}{M} ; \vartheta \right), \]

describing the amount of kinetic energy \( \Delta K \) exchanged by the two colliding particles (i.e., lost by the smaller one and acquired by the bigger one) relatively to the kinetic energy of the initially moving one (i.e., the smaller one). Figure 3.3 shows the functional dependence of \( \Xi \) on the particle mass ratio \( m/M \), for various values of the impact parameter \( b \) (through the deviation angle \( \vartheta(b) \)). Since \( \Xi \) is an even function of \( \vartheta \), the plot is representative of both equal and opposite signs for the electric charges. In the general case, we are interested into the region of low \( m/M \) ratio, because it is where our assumption holds. For any \( \vartheta \), energy transfer is observed to be a monotonically increasing function of the mass ratio. Since

\[ \Xi \left( \frac{m}{M} = 0; \vartheta \right) = 0, \quad \forall |\vartheta| \in [0, \pi], \]
energy exchange practically vanishes for collisions involving particles with very different mass, such as the electron-ion collision.

For the head-on collision case, where \( b = 0, |\vartheta| = \pi \) and the assumption that \( M \) be bigger than \( m \) is no longer necessary, we can also consider the whole \( m/M \in [0,1] \) range (the \( m > M \) case is superfluous, for the roles of particles are simply inverted). The rule according to which energy transfer and redistribution is more effective with particles of similar mass is confirmed and

\[
\Xi \left( \frac{m}{M} = 1; \vartheta = \pi \right) = 1 = \max_{\frac{m}{M} \in [0,1]} \left\{ \Xi \left( \frac{m}{M}; \vartheta = \pi \right) \right\}.
\]

### 3.3.4 Particle Energy Gain from the Electric Field

An important issue is what happens in between consecutive collisions as a consequence of the action of the electric field over charged particles. The basic fact is the work performed by the electric part of the Lorentz force, which transfers kinetic energy to the particles. The magnetic part of the Lorentz force can be rigorously disregarded, for it is orthogonal to the particle velocity and thus does not perform work.

We can investigate the energy transfer mechanism to charged particles from the electric field with the truly very simple model shown in Figure 3.4. For the sake of simplicity, we keep the argument in the classical framework, which is obviously incorrect, since the model describes an accelerating charge but cannot account for the energy thereafter radiated. Despite this incoherence, the model is sufficient to describe qualitatively the fundamental difference between the ion and the electron reaction, induced by their strongly unbalanced mass ratio. This will be the base for developing a non equilibrium model in the framework of the two-fluid description of the arc plasma (see §3.6.3), from whose implications we will derive a black box model for low voltage circuit breakers (see §5.4 and §5.6).

In any arc plasma, the charge carriers, that is, free electrons and positive ions, are accelerated by the electric field. Let \( \lambda \) be the mean free path between two subsequent collisions of an electron with heavy particles. The intensity of the force exerted by the electric field on charged particles is \( |Z_s| E e^- \), where \( E = |E| \) denotes the intensity of the electric field and where \( s = e \) is used to refer to an electron while \( s = i \) to refer to an ion. We assume that the electric field be constant over \( \lambda \), due to the slow rate of spatial and temporal variation of average electromagnetic quantities with reference to the micro scale of single particle dynamics.

Under the above assumptions, the dynamics of a charged particle is classi-
Figure 3.4: Simple model to study the energy transferred to charged particles by the electric field.

cally described by Newton’s law, i.e., by the ODE

\[ m_s \ddot{x}_s(t) = |Z_s| E e^- \quad s \in \{i, e\}, \]

where \( x_s(t) \) is the particle displacement from the original position, in the direction given by the imposed electric force. Newton’s law is immediately solved by time integrating twice, and reads

\[ x_s(t) = x_s(0) + v_s(0)t + \frac{|Z_s| E e^-}{2m_s} t^2 \quad s \in \{i, e\}, \]

where \( x_s(0) \) and \( v_s(0) \) are, respectively, the particle position and velocity at time \( t = 0 \). Since we are considering the particle displacement from the original position, then \( x_s(0) = 0 \). Assuming that each particle is originally motionless (the general case would be equally easy to solve, but with longer expressions and without real added value), i.e., \( v_s(0) = 0 \), the solution reads

\[ x_s(t) = \frac{|Z_s| E e^-}{2m_s} t^2 \quad s \in \{i, e\}. \]

The time required by an electron to be displaced by \( \Delta x_e = \lambda \) from its original position is

\[ \Delta t = \sqrt{\frac{2m_e \lambda}{|Z_e| E e^-}}. \]

During the same time interval, an ion is displaced by

\[ \Delta x_i = \frac{m_e}{m_i} \frac{|Z_i|}{|Z_e|} \Delta x_e \]

from its original position. The work performed by the electric force on each particle is obtained by taking the scalar product between force and displacement, and reads

\[ W_s = |Z_s| E e^- \Delta x_s \quad s \in \{i, e\}. \]
We can compare the work performed by the electric field on electrons and ions during the same time interval and find (recalling that $Z_e = -1$ and $Z_i \approx 1$ in low voltage arc plasma)

$$\Delta K_i \approx W_i = \frac{m_e}{m_i} \left( \frac{Z_i}{Z_e} \right)^2 W_e \approx \frac{m_e}{m_i} W_e \ll W_e \approx \Delta K_e,$$  \hspace{1cm} (3.14)

where $\Delta K_s$, $s \in \{e, i\}$, is the kinetic energy gain for particles of species $s$. The kinetic energy gain is equal to the work of the electric field up to the energy radiated away by the accelerating particle, which classically would be fully neglected. Inequality (3.14) shows that the work performed onto ions is way less (by a factor $m_e/m_i \ll 1$) than the work performed onto electrons. Neutrals, as such, are insensitive to the electric field but still receive a portion of the (very limited) kinetic energy gained by the ions. The mechanism of energy redistribution between ions and neutral is through collisions, which are non negligible for the mass similarity argument. Considering a multitude of collisions one finally gets

$$\Delta K_e \gg \Delta K_i \approx \Delta K_n \approx \Delta K_h.$$ \hspace{1cm} (3.15)

### 3.3.5 Lagrangian Approach

The approach utilized in the particle description of plasma is termed *Lagrangian* and it is characterized by describing the particles trajectories $x_{j,s}(t)$, for $j$ and $s$ ranging over the whole particle set. The Lagrangian approach is the immediate application of Newton’s laws of the mechanics for physical bodies, the latter being plasma particles (and for this reason it is also called *Newtonian*). In continuum mechanics, the Lagrangian approach is typical for solid mechanics, where a point $x$ of a solid body (possibly flexible) is fixed and described in its motion around the surrounding space.

### 3.4 The Kinetic Description

Plasma kinetic theory is very similar to gas kinetic theory, the most notable additional difficulty being that some of the particles are charged and electromagnetic fields, produced by the particles themselves or by external charges and/or currents, affects their motion. The kinetic description has some computational potentiality, though we shall only use it as a bridge toward a computational fluid approach.
3.4.1 Eulerian Approach

The point of view most suited for the kinetic description is different from the Lagrangian one. The approach termed *Eulerian* is used here and in the subsequent fluid description. The fundamental idea is to fix a point $x$ in space (not in the physical medium hosted in space) and to describe (in terms of velocity and other sensitive quantities) the parcels of fluid flowing through $x$ as time goes passing by, like when standing on a bridge and watching a river that flows underneath, keeping sight focused on fixed spot, e.g., of the bridge piles. The individual particles of fluid that can be found in $x$ are typically different from time to time, unless the fluid is (at least locally) motionless.

In continuum mechanics, the Eulerian approach is typical of fluid mechanics as an alternative to the Lagrangian one (even though the latter is most suited for some very particular fluid problems). The Lagrangian approach focuses on moving volumes, whilst the Eulerian approach focuses onto the moving fluid that happens to be inside a fixed volume of space. Even though the former is closer to intuition, the latter approach turns out to lead to a computationally more convenient formulation of flow equations and will be used hereafter. Since the two approaches describe the same physical phenomenon, there must of course be a matching point: the fluid velocity at a given time and fixed position (i.e., the Eulerian velocity) equals the velocity of the fluid parcel (i.e., the Lagrangian velocity) that occurs at that position and at that time.

3.4.2 Particle Density Functions

The particle approach is computationally unwieldy, because any particle is handled individually. To overcome the hyperbolic size of the problem resulting thereafter, a statistical approach is introduced. This amounts to losing part of the information (and increasing the entropy of the system, for Boltzmann H-theorem [76]). We retain the assumption that particles are described as points (see §3.3.1 for an introduction to the physical and engineering consequences). Particularly, in the 6-dimensional phase space $(x, v)$ a *phase space density* $f_s(x, v, t)$ (3.16)

\[ f_s(x, v, t) \]

is defined for any species $s$. Photons are an additional species that may be introduced to treat radiation in the same formal framework (see §3.8). The function $f_s$ is the particle density in the phase space, i.e., the 6-form

\[ f_s(x, v, t) \, dxdv \]
is the average number of particles of species $s$ whose spatial coordinates fall within $x$ and $x + dx$ and whose velocity components fall within $v$ and $v + dv$. Dimensionally, $[f_s] = [\ell^{-3}v^{-3}] = [v^3\ell^{-6}]$ and it is thus measured in $s^3m^{-6}$.

Integration over the velocity space yields the number density

$$n_s(x, t) := \int_{\mathbb{R}^3} f_s(x, v, t) \, dv$$

for the species $s$, which is the number of particle of species $s$ in a neighborhood of point $x$ at time $t$. The function $n_s$ is the particle density in the physical space (number of particles per unit volume). Dimensionally, $[n_s] = [\ell^{-3}]$ and it is thus measured in $m^{-3}$. The familiar mass density for the species $s$ is defined straightforwardly as

$$\rho_s(x, t) := n_s(x, t)m_s.$$  \hspace{1cm} (3.18)

Dimensionally, $[\rho_s] = [m\ell^{-3}]$ and it is thus measured in $kg\cdot m^{-3}$. By definition, $f_s \geq 0, \forall s$, and so also $n_s \geq 0$ and $\rho_s \geq 0, \forall s$.

The average velocity

$$u_s(x, t) := \frac{1}{n_s(x, t)} \int_{\mathbb{R}^3} v f_s(x, v, t) \, dv$$

for the species $s$ is very naturally defined as the weighted average of particle velocity, the weighting function being the phase space distribution function. The definition is well posed, for $f_s \geq 0, \forall s$.

### 3.4.3 Maxwell Equations

The charge density for the species $s$ is very naturally defined as

$$\varrho_s(x, t) := n_s(x, t)q_s,$$  \hspace{1cm} (3.20)

not to be confused with mass density $\rho_s$, defined by (3.18). Summing up over the different species, the total charge density

$$\varrho(x, t) := \sum_{s \in S} \varrho_s(x, t) + \varrho_{ext}(x, t)$$

is obtained, where $\varrho_{ext}(x, t)$ is the density of external charges. Since in our description of plasma all of the electric charges produced by air ionization have already been included into some $\rho_s$, external charges are located outside the plasma phase, e.g., in some solid part. Typically in circuit breakers the
two electric contacts, usually made of copper or some other good conducting material, when open behave as electrodes subject to the voltage difference imposed by the rest of the electric network. As a consequence, at any time one is a cathode, and a negative charge density is present in the solid phase bordering with plasma, and the other is an anode, and a positive charge density is present in the solid phase bordering with plasma.

In analogy with electric charge, the \textit{current density} for the species \(s\) is naturally defined as

\[ j_s(x, t) := n_s(x, t)q_s u_s(x, t). \]  

(3.22)

Adding up over the different species, the \textit{total current density}

\[ j(x, t) := \sum_{s \in S} j_s(x, t) + j_{\text{ext}}(x, t) \]  

(3.23)

is obtained, where \(j_{\text{ext}}(x, t)\) is the density of the external currents. Similarly to external charges, also external currents are located outside the plasma phase. Typically in circuit breakers external currents are inside the solid phase, that is, in the conductive path constituted by the electrodes (mobile and fixed contacts and the electric connections to the rest of the network). Particularly, in many low voltage circuit breakers the electric path comprises a solenoid, which is used as an electromechanical actuator to sense short-circuits and trip the contact opening mechanisms. Such a solenoid produces a magnetic field which sensibly affects the arc behavior.

Electric charge and current densities can be thought of as compactly supported functions, with a plasma phase contribution not overlapping with a solid phase contribution. Both contributions must be accounted for as sources of the electromagnetic field in the Maxwell equations. In differential form, Maxwell equations are the \textit{Gauss law}

\[ \nabla \cdot D = \rho, \]  

(3.24)

where \(D(x, t)\) is the electric flux density, \textit{Gauss law for magnetism}

\[ \nabla \cdot B = 0, \]  

(3.25)

where \(B(x, t)\) is the magnetic flux density, \textit{Faraday law}

\[ \nabla \times E = -\frac{\partial B}{\partial t}, \]  

(3.26)

where \(E(x, t)\) is the electric field, and \textit{Ampère law}

\[ \nabla \times H = j + \frac{\partial D}{\partial t} \]  

(3.27)
where \( \mathbf{H}(\mathbf{x}, t) \) is the magnetic field.

The electric field is related to the electric flux density by means of the dielectric constitutive equation

\[
\mathbf{D} = \varepsilon \mathbf{E},
\]

where \( \varepsilon = \varepsilon(\mathbf{x}, t) \) is the *permittivity* of the medium. In the general, anisotropic and inhomogeneous case, permittivity is a point wise defined tensor field. In the special case of low voltage circuit breakers we may assume the simplifying hypothesis of time independence, together with that of isotropic media, both for air plasma, plastic and metallic parts. Moreover, permittivity may also be assumed to be piecewise constant in the solid phase, that is, the solid parts of the circuit breaker may be conceived as an assembly of different materials, each one internally homogeneous and with a constant permittivity. The plasma phase is usually assumed to retain the permittivity of air, which is close to that of free space, that is, \( \varepsilon_0 \approx 8.854 \, 187 \, 817 \cdot 10^{-12} \, A^2 s^4 kg^{-1} m^{-3} \).

The magnetic field is related to the magnetic flux density by means of the magnetic constitutive equation

\[
\mathbf{B} = \mu \mathbf{H},
\]

where \( \mu = \mu(\mathbf{x}, t) \) is the *(magnetic) permeability* of the medium. In the general, anisotropic and inhomogeneous case, permeability is a point wise defined tensor field. In the special case of low voltage circuit breakers we may assume the simplifying hypothesis of isotropic media, both for air plasma, plastic and metallic parts. Moreover, permeability may also be assumed to be time independent and piecewise constant in the plasma phase as well as in solid parts, with the exception of ferromagnetic inclusions. Plastic and non ferromagnetic metals, such as copper or aluminium parts, may be conceived as an assembly of different materials, each one internally homogeneous and with a constant permeability. For ferromagnetic inclusions, the most notably example being the splitter plates, which are made of iron, a nonlinear relationship in between \( \mathbf{H} \) and \( \mathbf{B} \) must be assumed, with \( \mu = \mu(\mathbf{H}(\mathbf{x}, t)) \). The models usually employed in computational electromagnetism assume additionally that permeability only depends on the magnitude \( H \) of the magnetic field, that is,

\[
\mu = \mu(H(\mathbf{x}, t)).
\]

The nonlinear magnetic constitutive equation stemming from (3.30) accounts for magnetic saturation, which is an intrinsic and inescapable phenomenon associated with ferromagnets.

We explicitly point out that a model like (3.30) is nonlinear but still holonomic, which is a great simplification indeed. As a matter of fact, all ferromagnets exhibit a history dependent behavior, i.e., non holonomic, resulting
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Figure 3.5: Nonlinear magnetic constitutive law for typical ferromagnetic inclusions adopted in low voltage circuit breakers as splitter plates.

into the well-known hysteresis cycle (which is also frequency dependent). This all we neglect, since the iron materials used in the splitter plates of low voltage circuit breakers, as well as other notable ferromagnetic inclusions, have a thin hysteresis cycle. In the practice of computational electromagnetism this simplification is commonly adopted unless a ferromagnet is found with the explicit purpose of being a permanent magnet, and it is thus endowed with a pronounced hysteresis cycle.

The plasma phase is usually assumed to retain the magnetic permeability of air, which is very well approximated to that of free space, that is, $\mu_0 = 4\pi \cdot 10^{-7} H/m$. The same usually holds for all other non ferromagnetic components of low voltage circuit breakers.

We explicitly remind that permittivity and permeability of a given medium are related to the speed of light $c$ in that medium by

$$c = \frac{1}{\sqrt{\varepsilon\mu}}. \quad (3.31)$$

As a special case, (3.31) holds in vacuum. In the next developments we will omit the subscript 0 for the speed of light in vacuum.
3.4.4 Boltzmann Equation

We now come back to the mechanics of particles and deduce the fundamental equation of the kinetic approach. Collisional processes between particles of the same or of different species produce a change per unit time of the number of particles in a volume element of the 6-dimensional phase space, that is,

\[
\frac{d}{dt} f_s(x(t), v(t), t) = C_s
\]

where

\[
C_s = \sum_{s' \in S} C_{ss'}
\]

(3.32)
is an integral representation of collisional processes, extended to all possible kind of particles. Expanding the total derivative with reference to time, one immediately gets Liouville equation

\[
\frac{\partial f_s}{\partial t} + v \cdot \frac{\partial f_s}{\partial x} + \frac{q_s}{m_s} (E + v \times B) \cdot \frac{\partial f_s}{\partial v} = C_s.
\]

A more familiar form is obtained by plugging the Hamilton’s equations for the particle system, namely the equations of motion for the particles (kinematics and Newton’s law of dynamics), into Liouville equation, that is,

\[
\frac{\partial x}{\partial t} = v, \quad \frac{\partial v}{\partial t} = \frac{1}{m_s} f_s,
\]

where the Lorentz force

\[
f_s = q_s(E + v \times B)
\]
is acting upon the particles, in compliance with what assumed in (3.8). In the phase space \((x, v)\), position \(x\) and velocity \(v\) are independent variables, so that \((\partial/\partial x) \cdot v = 0\). Also, \((\partial/\partial v) \cdot f_s = 0\) when the Lorentz force is concerned, for \(\partial_i (v \times B)_j = \sum \partial_i (v_j B_k - v_k B_j) = 0\) for any even permutation of the indexes. Summing up, \((\partial/\partial x) \cdot \dot{x} = 0\) and \((\partial/\partial v) \cdot \dot{v} = 0\), that is, the phase space velocity \((\dot{x}, \dot{v})\) is divergence free, which implies that the flow in 6-dimensional phase space is incompressible and a 6-dimensional phase space (hyper-)volume is conserved along its trajectory (Liouville’s theorem for Hamiltonian systems [4]). This formally justifies the celebrated Boltzmann equation

\[
\frac{\partial f_s}{\partial t} + v \cdot \frac{\partial f_s}{\partial x} + \frac{q_s}{m_s} (E + v \times B) \cdot \frac{\partial f_s}{\partial v} = C_s.
\]

(3.33)

If the assumption is made, which we shall not, that \(C_s = 0\), then the kinetic equation is termed the Vlasov equation, or the collisionless Boltzmann equation.
When passing from the particle approach to the kinetic one, we left a set of ODE to end into a set PDE, which is an apparently absurd complication. But the phase space of the particle description is of dimension $6n$, with $n$ being the total number of particles, i.e., a huge number. Now, with Boltzmann equation, the phase space dimensionality is six fold, for each species (which is still a lot, but we will reduce it with the fluid and then magnetohydrodynamics approach). Mathematically, we left a conceptually nearly trivial, but monstrously large problem to get a more complex one, but with a first resemblance of practical tractability.

3.5 The Multi-Fluid Description

The computational approach to Boltzmann equation implies discretizing PDE over a 6-dimensional space, per each species. Though possible (e.g., by finite differencing), it remains a challenging problem even for nowadays computers (at least normal multi-core workstations largely available in the industrial research world).

To handle a smaller size problem we can describe plasma as a fluid. The fundamental underlying idea is to suitably take averages over velocity space, both of Boltzmann equation and of physical variables, and come back to the familiar 3-dimensional physical space. This amounts to treating a particle gas as a fluid continuum, with physical matter thought of as everywhere spread in space. We start with a \textit{multi-fluid} approach (or \textit{multi-species fluid}) where each chemical species is conceived as a different fluid, electrically charged in the case of electrons and ions and neutral otherwise. The many resulting continua are superimposed, coexisting and interacting by means of electromagnetic forces. From this brief introduction it is clear that, for the sake of convenience, we are introducing in our plasma model a number of abstractions, idealizations and, from some point of view, absurdities. Despite this, the concept of continuum is very natural for us, because it is the level of reality we perceive with our senses.

3.5.1 The General Transport Equation

From the mathematical and physical standpoint, the passage from kinetic to fluid theory corresponds to the passage from Boltzmann equation to the \textit{general}
transport equation, which, for a generic transported quantity\(^2\) \(pg\), reads

\[
\frac{\partial pg}{\partial t} + \nabla \cdot (pgu_s) - \nabla \cdot \Gamma \nabla g = S_g.
\] (3.34)

We now analyze the structure of this equation and describe its physical interpretation, for this will be the way in which we shall express all physical conservation laws.

In the Eulerian approach, the governing equations of a fluid are deduced with reference to an infinitesimal control volume \(d\mathbf{x}\) fixed in space and traversed by a fluid in motion (including the special case when the fluid is at rest). For the sake of simplicity we shall make the extrinsic choice of an orthogonal Cartesian frame, i.e., we shall refer to a cubic volume \(d\mathbf{x} = dx_i \, dx_j \, dx_k\), though the equations deduced hereafter are liable to be immediately cast in intrinsic form.

Given a generic quantity \(\rho(x,t)g(x,t)\) associated to the flow like, e.g., mass density, or momentum, etc., the convective flux \(\Phi^c_g\) of \(g\) through an infinitesimal surface \(dA\) is defined as

\[
\Phi^c_g(x,t) := \rho(x,t)g(x,t)u_s(x,t) \cdot n \, dA,
\] (3.35)

where \(u_s\) is the average flow velocity for species \(s\) and \(n\) is the unit normal vector to \(dA\). When dealing with the boundary of the control volume, we shall always consider outward normals. With reference to the Cartesian control volume, we choose the \(e_i\) direction and a surface \(dx_j \, dx_k\) orthogonal to \(n = e_i\) and centered in \(x\). The relevant convective flux is \(\Phi^c_g(x,t) = \rho(x,t)g(x,t)u_{si}(x,t)dx_j \, dx_k\). Considering spatial dependence and truncating at first order the Taylor series expansion of \(\Phi^c_g\), one gets (up to higher order infinitesimals) the convective flux through the opposite face of the cubic control volume as

\[
\Phi^c_g(x + e_i \, dx_i, t) \approx \left( \rho(x,t)g(x,t)u_{si}(x,t) + \left. \frac{\partial \rho gu_{si}}{\partial x_i} \right|_{(x,t)} \right) dx_j \, dx_k.
\]

Thus, up to higher order infinitesimals, the net outflow in the selected direction is obtained as \(\Phi^c_g(x + e_i \, dx_i, t) - \Phi^c_g(x, t) \approx \partial_i (\rho gu_{si})dV\) (if this quantity is positive, then we actually have an outflow, for the \(x_i\)-axis points outward of the control volume in case of \(\Phi^c_g(x + e_i \, dx_i, t)\) and inward in case of \(\Phi^c_g(x, t)\)). Accounting for the other two Cartesian directions, the total net convective outflow per unit volume is

\[
\sum_{i=1}^{3} \frac{\partial \rho gu_{si}}{\partial x_i} = \nabla \cdot (\rho gu_s).
\]

\(^2\)The explicit presence of mass density \(\rho\) is only for the sake of convenience
Following the same notation as for the convective flux, the *diffusive flow* \( \Phi_g^d \) of \( g \) through an infinitesimal surface \( dA \) is defined as

\[
\Phi_g^d(x, t) := (-\Gamma(x, t) \nabla g(x, t)) \cdot n \, dA,
\]

(3.36)

where \( \Gamma(x, t) \) is the *diffusivity tensor*. In an isotropic medium, the second order diffusivity tensor reduces to the identity tensor multiplied by a scalar property, the familiar diffusivity. Handling the diffusive flux the same way as the convective flux, the net outflow in direction \( i \) is obtained as \( \Phi_g^d(x + e_i dx_i, t) - \Phi_g^d(x, t) \approx \partial_i (-\Gamma \nabla g)_i dV \). Accounting for the other two Cartesian directions, the total net diffusive outflow per unit volume is

\[
\sum_{i=1}^{3} \frac{\partial}{\partial x_i} (-\Gamma \nabla g)_i = -\nabla \cdot \Gamma \nabla g.
\]

One can now look at the general transport equation (3.34) as the global balance for the transported quantity \( g \) over the control volume: the time variation of \( g \), i.e., the *storage term* \( \partial \rho g / \partial t \) (positive in case of an increment), equals what is created inside the control volume, i.e., the *source term* \( S_g \) (dubbed a *sink* when negative), reduced by the outflow, i.e., the *convective (or transport) term* \( \nabla \cdot (\rho g u_s) \) and the *diffusive term* \( -\nabla \cdot \Gamma \nabla g \).

### 3.5.2 Kinetic averages

Some mathematical formalism is here introduced, that will allow a great simplification in the theory, based on the idea of averaging quantities over the velocity space.

**Kinetic Average Operators**

The average velocity \( u_s \) of a species \( s \) is a special case of the concept of the *local velocity space average*, according to the species \( s \), of a function \( \Psi(\mathbf{v}) \) of velocity. Precisely, the \( \langle \cdot \rangle_s \) operator, for the species \( s \), is defined as

\[
\langle \Psi(\mathbf{v}) \rangle_s := \frac{1}{n_s(x, t)} \int_{\mathbb{R}^3} \Psi(\mathbf{v}) f_s(x, \mathbf{v}, t) \, d\mathbf{v}.
\]

(3.37)

From the comparison of (3.19) and (3.37), it follows immediately that

\[
u_s = \langle \mathbf{v} \rangle_s, \quad \forall s \in \mathcal{S}.\]
From the definition (3.37) and the linearity of the integral operator, the \( \langle \cdot \rangle_s \) operators are also linear, i.e.,
\[
\langle c_1 \Psi_1(v) + c_2 \Psi_2(v) \rangle_s = c_1 \langle \Psi_1(v) \rangle_s + c_2 \langle \Psi_2(v) \rangle_s, \quad \forall c_1, c_2 \in \mathbb{R}, \, \forall s \in S. \tag{3.38}
\]

### Average and Fluctuating Velocity Decomposition

It is very useful to distinguish between the average (point wise) velocity of the particles of a species and the chaotic velocity fluctuations around such an average value. For this purpose the *velocity fluctuation*
\[
\tilde{v}_s := v - u_s \tag{3.39}
\]
for the species \( s \) is defined. Due to the linearity property (3.38) of local average operators, velocity fluctuations have null average, that is,
\[
\langle \tilde{v}_s \rangle_s = 0, \quad \forall s \in S. \tag{3.40}
\]
Definition (3.39) induces the *velocity decomposition*
\[
v = u_s + \tilde{v}_s \tag{3.41}
\]
for the species \( s \).

In the following, when dealing with the momentum balance equation, we shall need the tensor product \( v \otimes v \) of velocity, which can be represented as a matrix quantity whose \( ij \)-th entry is defined as \( (v \otimes v)_{ij} := v_i v_j \). From the definition, it follows that \( a \otimes b = (b \otimes a)^T, \forall a, b \in \mathbb{R}^n, \, n \in \mathbb{N} \). Velocity decomposition (3.41) allows decomposing the tensor product of velocity as
\[
v \otimes v = u_s \otimes u_s + u_s \otimes \tilde{v}_s + \tilde{v}_s \otimes u_s + \tilde{v}_s \otimes \tilde{v}_s. \tag{3.42}
\]

Averaging (3.42) over velocity space, one gets
\[
\langle v \otimes v \rangle_s = u_s \otimes u_s + \langle \tilde{v}_s \otimes \tilde{v}_s \rangle_s, \tag{3.43}
\]
because \( \langle u_s \otimes \tilde{v}_s \rangle_s = u_s \otimes \langle \tilde{v}_s \rangle_s = 0 \), where \( 0 \) is here the null \( 3 \times 3 \) tensor, for the linearity property (3.38), and for (3.40), and analogously its transpose \( \langle \tilde{v}_s \otimes u_s \rangle_s = 0 \). It will also be useful to remind that the square of the velocity magnitude can be expressed as the scalar product \( v^2 = v \cdot v \), from which and from the velocity decomposition (3.42) it immediately follows that
\[
v^2 = u_s^2 + 2 u_s \cdot \tilde{v}_s + \tilde{v}_s^2. \tag{3.44}
\]

The square velocity magnitude also coincides with the trace of the tensor product of the velocity vector by itself, i.e., \( v^2 = \text{tr}(v \otimes v) \).
Pressure

The pressure tensor for the species $s$ is defined as

$$P_s(x,t) := \rho_s \langle \tilde{v}_s \otimes \tilde{v}_s \rangle_s = m_s \int_{\mathbb{R}^3} \tilde{v}_s \otimes \tilde{v}_s f_s(x,v,t) \, dv. \quad (3.45)$$

Accounting for the pressure tensor, identity (3.43) may be rewritten as

$$\rho_s \langle v \otimes v \rangle_s = \rho_s u_s \otimes u_s + P_s. \quad (3.46)$$

As usual in solid and fluid mechanics, the pressure tensor is decomposed into a spherical (or hydrostatic) part $p_s(x,t)I$ and a deviatoric part $\Pi_s(x,t)$ as

$$P_s = p_s I + \Pi_s. \quad (3.47)$$

In (3.47) $I$ is the unit (or identity) tensor and $p_s$ is the hydrostatic pressure for the species $s$, defined as one third of the trace of the pressure tensor, i.e.,

$$p_s := \frac{1}{3} \text{tr} P_s = \frac{1}{3} \sum_{i=1}^{3} (P_s)_{ii} = \frac{1}{3} m_s \int_{\mathbb{R}^3} \tilde{v}_s^2 f_s(x,v,t) \, dv = \frac{1}{3} \rho_s \langle \tilde{v}_s^2 \rangle_s. \quad (3.48)$$

We remind that the deviator of a tensor is simply defined as what is left behind after the spherical component is removed and it is thus traceless. Due to the definition (3.48) of hydrostatic pressure, to identity (3.44) and to (3.40), the identity

$$\frac{1}{2} \rho_s \langle v^2 \rangle_s = \frac{1}{2} \rho_s u_s^2 + \frac{3}{2} p_s \quad (3.49)$$

follows straightforwardly.

Our purpose is now to give back a physical sense to the concept of pressure, defined by (3.48) and so far intended in a merely mathematical sense, and to use it in order to introduce temperature as a new quantity when describing arc plasma. To accomplish to this task, we imagine gas particles bouncing against a plane wall located in point $x$ (see Figure 3.6, left). If a given particle of species $s$ with velocity $v$ collides with an angle $\theta$ to the wall unit normal $\hat{n}$ and bounces back according to elastic collision theory (i.e., with the same velocity magnitude and opposite angle in the plane containing $v$ and $\hat{n}$), then the momentum received by the wall is $2m_s v \cos \theta \hat{n}$.

We want to compute the number of particles bouncing exactly this way in a time interval of duration $dt$ and against a wall surface element $dS$. The particles we are interested in must be close enough to bounce against the wall. Only those are good which are located in a prismatic volume element of base
3.5. THE MULTI-FLUID DESCRIPTION

Figure 3.6: Physical interpretation of pressure: momentum exchange between a gas particle and a wall (left picture); volume of gas whose particles collide against a wall infinitesimal element $dS$ in infinitesimal time $dt$ under angle $\theta$ (central picture); half solid angle (right picture).

$dS$, leaned with angle $\theta$ and with height $v \cos \theta \, dt$ (see Figure 3.6, center). So we start considering a quantity $n_s v \cos \theta \, dtdS$. As regards velocity space, we have a requirement on velocity magnitude and one on velocity direction. If $y(v)$ is the probability density function of velocity magnitude, than the fraction of particles with the right velocity is obtained by multiplication by $y(v) \, dv$. Assuming an isotropic velocity distribution, the fraction of particles with angle $\cos \theta$ is produced by multiplication by the ratio of the solid angle portion $2 \pi \sin \theta \, d\theta$ (see Figure 3.6, right) and the whole solid angle $4 \pi$. In the end, the number of good particles is $n_s v \cos \theta y(v) \frac{1}{2} \sin \theta \, dv \, d\theta \, dtdS$.

To get the momentum $dq$ transferred to a wall element $dS$ in the interval $dt$ we need accounting for the momentum transferred from each particle of the above type and then we need integrating over half velocity space (only those directions pointing toward the wall), obtaining

$$dq = -n_s m_s \cdot \int_0^{\infty} y(v) v^2 \, dv \cdot \int_0^{\pi/2} \cos^2 \theta \sin \theta \, d\theta \cdot dtdS \cdot \hat{n} = -\frac{1}{3} \rho_s \langle v^2 \rangle_s dtdS \cdot \hat{n}.$$  

Finally, the magnitude of the force acted upon the wall element $dS$ by the gas particles of species $s$ is $dF_s = dq / dt$ and thus the pressure, or force per unit surface, is $-p_s \hat{n} = dF_s / dS$, that is, exactly the relation

$$p_s = \frac{1}{3} \rho_s \langle v^2 \rangle_s$$  

(3.50)

that we found in (3.48) and which is now physically interpreted.
CHAPTER 3. LOW VOLTAGE ARC PHYSICS

Temperature

The concept of temperature is very natural when describing electric arc plasma at the engineering macro scale, particularly when expressing electrical conductance as a strongly temperature dependent function. Also, arc temperature is one of the key factors during an electric circuit interruption, decreeing its success or failure. Temperature is a physical quantity which is not applicable at the particle level description and is introduced to coarser physical scales through kinetic gas theory. We formally define the temperature $T_s$ of the species $s$ as

$$T_s := \frac{p_s}{kn_s}.$$  \hspace{1cm} (3.51)

From the definition, it immediately follows the celebrated relation

$$\frac{1}{2}m_s\langle v^2 \rangle_s = \frac{3}{2}kT_s.$$  \hspace{1cm} (3.52)

From Boltzmann equipartition theorem we know that kinetic energy is, on average, evenly distributed over the degrees of freedom of the particle, each one receiving a $\frac{1}{2}kT_s$ contribution. Since we have developed the theory with a point shaped model of particle (see §3.3.1), thus endowed with three degrees of freedom, this result is coherent with the premises.

Heat Flux

The heat flux carried by particle species $s$ is defined as

$$q_s(x, t) := \frac{1}{2} \rho_s \langle \tilde{v}_s \tilde{v}_s^2 \rangle_s = \int \frac{1}{2} m_s \tilde{v}_s^2 \tilde{v}_s f_s(x, v, t) \, dv.$$  \hspace{1cm} (3.53)

Dimensionally, $[q_s] = [mt^{-2}]$, that is, it is a surfacic power flux, whence the denomination. The units of heat flux are thus $W/m^2$. Thanks to the above definitions, the identity

$$\frac{1}{2} \rho_s \langle v^2 \rangle_s = \left( \frac{1}{2} \rho_s u_s^2 + \frac{3}{2} p_s \right) u_s + P_s u_s + q_s$$  \hspace{1cm} (3.54)

is produced, which will be extremely useful in simplifying the energy balance equation. The deduction of (3.54) relies on long but simple algebraic manipulations. The only non obvious passage is $\rho_s \langle \tilde{v}_s (u_s \cdot \tilde{v}_s) \rangle_s = P_s u_s$. A simple way to prove it is to work component wise and notice that $\rho_s \tilde{v}_s (u_s \cdot \tilde{v}_s)_i = \rho_s \tilde{v}_s u_s \cdot \tilde{v}_s = \sum_{j=1}^3 \rho_s \tilde{v}_{si} \tilde{v}_{sj} u_{sj} = \sum_{j=1}^3 (\rho_s \tilde{v}_s \otimes \tilde{v}_s)_{ij} (u_s)_j = ((\rho_s \tilde{v}_s \otimes \tilde{v}_s) u_s)_i$, whence (3.54) immediately follows.
3.5. **THE MULTI-FLUID DESCRIPTION**

**Moments of Boltzmann Equation**

The basic conservation principles of mechanics are obtained by taking the moments of Boltzmann equation (3.33). First, we notice that

**Lemma 3.1.** Multiplying Boltzmann equation (3.33), \( \forall s \in S \), by \( \Psi(v) \) and integrating over the velocity space, that is,

\[
\int_{\mathbb{R}^3} \left( \frac{\partial f_s}{\partial t} + v \cdot \frac{\partial f_s}{\partial x} + \frac{q_s}{m_s} (E + v \times B) \cdot \frac{\partial f_s}{\partial v} \right) \Psi(v) \, dv = \int_{\mathbb{R}^3} C_s \Psi(v) \, dv,
\]

one gets

\[
\frac{\partial}{\partial t} \langle n_s \Psi \rangle_s + \nabla \cdot (n_s (v \Psi)) - \frac{n_s q_s}{m_s} \left[ \frac{\partial}{\partial v} \left( \frac{\partial \Psi}{\partial v} \right) \right]_{s} + \left\langle v \times B \cdot \frac{\partial \Psi}{\partial v} \right\rangle_s = \int_{\mathbb{R}^3} C_s \Psi(v) \, dv. \tag{3.55}
\]

**Proof.** The first two terms under the integral sign on the l.h.s. are easily transformed by swapping the integral itself, which is over velocity, with the time and space derivatives, respectively, and accounting for (3.37). The transformation of the term with Lorentz force additionally requires the application of Green’s formulae. One can argument on a bounded volume in velocity space, e.g., a spherical one with radius \( v = |v| \), then take the limit when the domain becomes unbounded and account for

\[
\lim_{v \to \infty} \Psi(v) f_s(x, v, t) = 0.
\]

The latter can be justified by assuming that \( f_s \) be a rapidly decreasing function of velocity, so to annihilate the possible divergence of \( \Psi \). In the following, \( \Psi \) will be an algebraic function of velocity, so that the limit holds if, e.g. but not in so far as, \( f_s \) is Maxwellian distribution, which is exponentially (i.e., more than algebraically) decreasing. Finally, the identity

\[
\frac{\partial}{\partial v_i} (v \times B)_i = \sum j \frac{\partial}{\partial v_i} (v_j B_k - v_k B_j) = 0
\]

has to be remembered when handling the magnetic part of Lorentz force. \( \square \)

In the next sections we shall make use of Lemma 3.1 to generate the first moments of Boltzmann equation.

**3.5.3 Mass Balance**

First we choose \( \Psi = 1 \) in Lemma 3.1, that means, we take the zero-th moment of Boltzmann equation, and obtain, \( \forall s \in S \), the particle balance

\[
\frac{\partial}{\partial t} n_s + \nabla \cdot (n_s u_s) = \sum_{s' \in S} \int_{\mathbb{R}^3} C_{ss'} \, dv. \tag{3.56}
\]
In order to simplify the particle balance to a more gentle equation, we first
include the collisional terms, \( \forall s \in \mathcal{S} \), into the symbol
\[
S_s := \sum_{s' \in \mathcal{S}} \int_{\mathbb{R}^3} m_s C_{ss'} \, dv.
\] (3.57)

Then, by multiplication by the (constant) particle mass \( m_s \), one gets, \( \forall s \in \mathcal{S} \), the mass balance
\[
\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s u_s) = S_s.
\] (3.58)

Mass balance (3.58) has the form of the general transport equation (see §3.5.1). The physical meaning is that the mass source term on the r.h.s. equals
the sum of mass variation in time (storage term) and mass outflow (convective
term). In case ionizations, recombinations and particle addition due to material
ablation are neglected, then \( C_s = \sum_{s' \in \mathcal{S}} C_{ss'} = 0 \) since particle collisions
do not give rise to any inter species mass transfer. In this case, it follows
\( S_s = 0, \forall s \in \mathcal{S} \). In case only material ablation is neglected, with ionization
and recombination possibly accounted for, then \( S_s \) needs not to be vanishing,
but still \( \sum_{s \in \mathcal{S}} S_s = 0 \), since the overall mass balance must be satisfied.

### 3.5.4 Momentum Balance

We now choose \( \Psi = m_s v_i \), for \( i \in \{1, 2, 3\} \), in Lemma 3.1, that means, we take
the first order moment of Boltzmann equation, and obtain, \( \forall s \in \mathcal{S} \), the \( i \)-th momentum balance
\[
\frac{\partial}{\partial t} \left(n_s m_s u_{si}\right) + \nabla \cdot \left(n_s m_s (v v_i)_s\right) - n_s q_s \left(E_i + (u_s \times B)_i\right)
\] (3.59)
\[
= \sum_{s'} \int_{\mathbb{R}^3} m_s v_i C_{ss'} \, dv.
\]

Also in this case one wants to reduce to a more gentle equation. The vector
form of momentum balance is simpler to handle. For the sake of compactness
we collect collisional terms into a single symbol by setting, \( \forall s \in \mathcal{S} \),
\[
R_s := \sum_{s' \in \mathcal{S}} \int_{\mathbb{R}^3} m_s \tilde{v}_s C_{ss'} \, dv.
\] (3.60)

This enables rewriting (3.59) as
\[
\frac{\partial}{\partial t} (\rho_s u_s) + \nabla \cdot (\rho_s (v \otimes v)_s) - q_s (E + u_s \times B) = R_s + u_s S_s.
\] (3.61)
The first addendum on the l.h.s. is simply the transcription in vector form of $\frac{\partial}{\partial t}(n_s m_s u_{si})$, accounting for (3.18). We remind that the presence of $u_s$ was due to averaging over velocity space. Similar considerations hold for the second addendum, which is simply the vector form of $\nabla \cdot (n_s m_s \langle v_{vi} \rangle_l)$, with the introduction of the tensor product of velocity by itself. This is the notorious convective term, which is quadratic and makes the equations of fluid mechanics nonlinear and thus so problematic. In producing the vector form of the Lorentz force term, (3.20) has been accounted for. The r.h.s. follows trivially from (3.60) and (3.57).

Finally, the momentum balance for species $s$ reads

$$\frac{\partial}{\partial t}(\rho_s u_s) + \nabla \cdot (\rho_s u_s \otimes u_s) = -\nabla p_s - \nabla \cdot \Pi_s + g_s(E + u_s \times B) + R_s + u_s S_s,$$

(3.62)

where the decomposition (3.43) of the convective term has been introduced, together with the splitting of the pressure tensor into the spherical and the deviatoric part.

Momentum balance (3.62) has the form of the general transport equation (see §3.5.1). The physical meaning is that the momentum source term on the r.h.s., that is, the forces, equals the sum of momentum variation in time (storage term) and momentum outflow (convective term). External forces (to species $s$) include, from left to right at the r.h.s., pressure gradients, viscous dissipations, Lorentz force, collisional momentum exchange with other species and momentum gain or loss due to ionizations/recombinations and material ablation. If ionizations, recombinations and material ablations are not accounted for, then $S_s = 0$, $\forall s \in S$. In case only material ablation is neglected, with ionizations and recombinations possibly accounted for, then $R_s$ need not be vanishing, but still $\sum_{s \in S} R_s = 0$, since the overall momentum balance must be satisfied.

### 3.5.5 Energy Balance

Finally we choose $\Psi = \frac{1}{2} m_s v^2$ in Lemma 3.1, that means, we take the second order moment of Boltzmann equation, and obtain, $\forall s \in S$, the energy balance

$$\frac{\partial}{\partial t} \left( \frac{1}{2} n_s m_s \langle v^2 \rangle_s \right) + \nabla \cdot \left( \frac{1}{2} n_s m_s \langle vv \rangle_s \right) - n_s q_s u_s \cdot E = \sum_{s' \in S} \int_{R^3} \frac{1}{2} m_s v^2 C_{ss'} d\mathbf{v}.$$

(3.63)
Some comments are needed with reference to the Lorentz force term. When taking the velocity space average of $\partial \Psi / \partial \mathbf{v}$ one may profitably notice that

$$\frac{1}{2} \frac{\partial v^2}{\partial v_i} = \frac{1}{2} \frac{\partial}{\partial v_i} (v_i^2 + v_j^2 + v_k^2) = v_i \Rightarrow \frac{1}{2} \frac{\partial v^2}{\partial \mathbf{v}} = \mathbf{v}.$$  

Averaging over velocity, $\mathbf{u}_s$ is produced, which immediately explains the electric portion. The magnetic portion disappears because of the identity $(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$. In other words, since Lorentz force acts orthogonally to particles velocity, its work identically vanishes and no contribution is given in energizing the particles, a very well-known and intuitive fact.

Once again, we collect collisional terms under the single symbol

$$Q_s := \sum_{s' \in S} \int \frac{1}{2} m_s \tilde{v}_s^2 C_{ss'} d\mathbf{v},$$

so that the energy balance equation for the specie $s$ may be simplified to

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_s u_s^2 + \frac{3}{2} p_s \right) + \nabla \cdot \left( \frac{1}{2} \rho_s u_s^2 \mathbf{u}_s + \frac{3}{2} \rho_s \mathbf{u}_s + \mathbf{P}_s \cdot \mathbf{u}_s + \mathbf{q}_s \right) = j_s \cdot \mathbf{E} + u_s^2 S_s + Q_s + \mathbf{u}_s \cdot \mathbf{R}_s. \tag{3.64}$$

When passing from (3.63) to (3.64), we made use of identities (3.49) and (3.54) to handle the first and second term, respectively, and we have introduced mass and current density. The meaning of (3.64) as an energy balance equation in the form of a generalized transport equation is currently not explicitly apparent but will disclose in §3.7, as soon as Poynting theorem has been used to rearrange terms.

### 3.5.6 System Closure

Before going forward with the theory, a moment of reflection is appropriate. When dealing with the particle description we had to solve Newton’s law for a very large multitude of particles and, apart from the obvious computational difficulties, the system was closed, meaning that nothing else was conceptually needed to solve it. Then we drifted to the kinetic description, characterized by Boltzmann equation. Once again the system was closed, for Boltzmann equation already contains all the physical information, but the problem is now the setting in the phase space.

The progression to the fluid description is obtained by means of taking moments of Boltzmann equation with reference to functions $\Psi$ of velocity and
integrating over velocity space; see Lemma 3.1. This amounts to passing from
the strong formulation of Boltzmann PDE, to hold point wise in time, space
and velocity space, to a weak (or variational) formulation, to hold point wise in
time and space but only in a weighted sense in velocity space; see §4.3.2. If one
looks at this a bit fancily, we left phase space but we are recovering velocity
dependence in an averaged way. Functions $\Psi$ played the role of trial functions
in the transition from the strong to the weak formulation. If one pays attention,
we first took a constant function of velocity, namely $\Psi = 1$, then we took three
linear functions of velocity, namely $\Psi = m_i v_i$, $i \in \{1, 2, 3\}$, and finally we
took one quadratic function of velocity, namely $\Psi = 1/2m_s v^2$. Apart from
the actual expression of trial functions, which are purposely chosen to yield
the conservation principles of mechanics, we adopted a set of polynomials of
increasing degree. Of course the polynomials used are linearly independent
for the degree argument, but, since we stopped at degree 2, it is obvious that
completeness of the trial space spanned by functions $\Psi$ cannot hold in any way,
nor, as consequence, the system may be closed.

That the system is not closed, one may understand immediately also by
simply looking at the final set (3.58), (3.62) and (3.64) that we have obtained.
The particle, or mass balance (3.58), describes the zero-th order moment of $f_s$
and requires the knowledge of average velocity $\mathbf{u}_s = \langle v \rangle_s$, which is the first
order moment of $f_s$. The evolution of the latter is described by momentum
balance (3.62), where the knowledge of the second order moment $\langle v \otimes v \rangle_s$ is
required. In its turn, the energy balance (3.64) requires the third order moment
and, since in the transport term velocity appears as a multiplying factor, an
infinite hierarchy is produced. Truncation must be performed, sooner or later,
and something additional, inside which only lower order moments are present,
is required to close the system.

State equations (one per chemical species in the multi fluid approach, or just
one in the single fluid approach, soon to come), as well an explicit expression
for $\Pi_s$ by means of viscous stresses are used to close the system phenomenolog-
ically. We shall come back on this issue when dealing with air plasma approxi-
imated as a single fluid, but the problem and the solution are fairly general and
may be extended also to multi-species fluids. The above argument also explains
why many references on plasma physics (e.g., [48]) chop the hierarchy at the
level of momentum balance, that is, one step before us, without mentioning the
energy balance. In the end it is a different, but still conceptually reasonable
and lawful approximation, provided that energy is not of interest (which is not
our case, since Joule heating is exactly what keeps the electric arc burning,
when abundant, or decrees its quenching, when scarce).
3.6 The Two-Fluid Description

In the multi-fluid description, a relatively large set of PDE must be solved for each species, and such sets are coupled through collisional terms. The resulting mathematical problem is therefore rather big and a simplification is required to obtain something more tractable from the computational standpoint. A first step is the two-fluid approach (or two-species fluid), where a unique, equivalent species is defined for the heavy particles (ions and neutrals). The second species is constituted by electrons, which are way lighter mass particles with reference to ions and neutrals, as already stated in §3.3.2.

Likewise, it would also be possible to stop one step before and define a three-species fluid accounting for electrons, ions and neutrals. The conceptual and mathematical operations required are analogous in each case. We will refer to the two-fluid for it is sufficient for a comprehension of some fine features inside low voltage circuit breakers, namely those pertaining to the non equilibrium physics. This will also constitute the theoretical base when defining a black box model for low voltage arcs close to extinction.

3.6.1 Equivalent Particle Species

In this section we see how to define and work with equivalent species. First, the set $\mathcal{S}$ of all particle species is partitioned into the singleton of electrons and the set $\mathcal{S}_h$ of heavy particles, that is,

$$\mathcal{S} = \{e^-\} \cup \mathcal{S}_h,$$

where, obviously, $\{e^-\} \cap \mathcal{S}_h = \emptyset$. The set $\mathcal{S}_h$ includes the set of ions $\mathcal{S}_i$ and the set of neutrals $\mathcal{S}_n$, or

$$\mathcal{S}_h = \mathcal{S}_i \cup \mathcal{S}_n,$$

where, obviously, $\mathcal{S}_i \cap \mathcal{S}_n = \emptyset$.

The heavy particle equivalent particle density

$$n_h := \sum_{s \in \mathcal{S}_h} n_s$$  \hfill (3.65)

is naturally defined by taking the sum over all of the heavy particle species densities. The heavy particle equivalent mass density

$$\rho_h := \sum_{s \in \mathcal{S}_h} \rho_s = \sum_{s \in \mathcal{S}_h} n_s m_s$$  \hfill (3.66)
is defined likewise by taking the sum over all of the heavy particle species mass densities. By inspection of (3.21) and (3.23), one immediately sees that electric charge and current are handled the same way (i.e., by summation) when dealing with equivalent species.

On the other hand, the heavy particle equivalent fluid velocity

\[ u_h := \frac{1}{\rho} \sum_{s \in S_h} \rho_s u_s \]  

(3.67)

is defined as the weighted average velocity over all of the particle species average velocities, the weighting function being the mass density. The definition is well posed, for \( \rho_s \geq 0, \forall s \in S \supset S_h \).

The heavy particle equivalent pressure tensor is defined similarly to (3.45) as

\[ P_h := \sum_{s \in S_h} P^m_s, \]  

(3.68)

where

\[ P^m_s := \rho_s \langle (v - u_h) \otimes (v - u_h) \rangle_s, \quad \forall s \in S_h. \]  

(3.69)

The relation with the pressure tensors of the plasma constituting species is expressed by the following

**Lemma 3.2.** The heavy particle pressure tensor is given by

\[ P_h = \sum_{s \in S_h} (P_s + \rho_s u_s \otimes u_s) - \rho_h u_h \otimes u_h. \]  

(3.70)

**Proof.** With a very common argument, the decomposition \( v - u_h = (v - u_s) - (u_s - u_h) = \tilde{v}_s - (u_s - u_h) \) is introduced and plugged into the r.h.s. of (3.69). From this point on, it is simply a matter of carrying out long but conceptually harmless algebraic manipulations. Each direct term only involving \( \tilde{v}_s \) yields a \( P_s \) contribution, as found on the r.h.s. of (3.70). Since both \( u_s \) and \( u_h \) are independent of the fluctuating part of velocity, they are left unchanged by the kinetic average operators. Then each mixed term vanishes, for \( \langle \tilde{v}_s \otimes (u_s - u_h) \rangle_s = \langle \tilde{v}_s \rangle_s \otimes (u_s - u_h) \) and \( \tilde{v}_s \) has null average for (3.40). A similar argument holds for the transpose term. As regards the \( \sum_s \rho_s (u_s - u_h) \otimes (u_s - u_h) \) contribution, we go on expanding, A \( \sum_s \rho_s u_s \otimes u_s \) term, as found on the r.h.s. of (3.70), is produced by all of the direct terms in \( u_s \). Due to (3.66), the sum of the direct terms in \( u_h \) yields \( \sum_s \rho_s u_h \otimes u_h = \rho_h u_h \otimes u_h \). Due to (3.67), the sum of the mixed terms yields \(-\sum_s \rho_s (u_s \otimes u_h + u_h \otimes u_s) = -2 \rho_h u_h \otimes u_h \). Summing up, (3.70) holds.

The heavy particle equivalent hydrostatic pressure

\[ p_h := \frac{1}{3} \sum_{i=1}^{3} (P_h)_{ii} \]  

(3.71)
is defined as one third of the trace of the relevant pressure tensor, as usual. Mass, momentum and energy conservation equation are obtained by summation over heavy species. For the sake of clarity, we show the form taken by mass balance for heavy particles, that reads

\[ \frac{\partial \rho_h}{\partial t} + \nabla \cdot (\rho_h \mathbf{u}_h) = 0, \]  

(3.72)

where there is no collisional source term since ions are produced out of neutrals, that is, out of other heavy particles, so that there is neither net production nor consumption. Momentum and energy conservation balance are treated likewise from their counterpart applicable on a species basis.

There is nothing new to say about the electron gas, since the light particle class only includes one species and equivalent quantities simply coincide with the relevant ones defined at the level of the unique species. In conclusion, mass, momentum and energy conservation equations for the electron gas are written the same as in multi-fluid plasma description.

### 3.6.2 Saha Equation

When a multi-fluid model (particularly, a two-fluid model) has to be solved, the concentration of the different species must be accounted for. This implies introducing some relations describing the ionization/recombination equilibrium, which is a balance between the energetic content of colliding particles and ionization energies. The former is intrinsically related to the concept of temperature, whilst the latter can be computed with quantum mechanics.

Not surprisingly, the governing equation, termed Saha ionization equation (also known as Saha-Langmuir equation of thermal ionization [35], or simply Saha equation), combines ideas of statistical and quantum mechanics. The original version was formulated in the 1920’s by Megh Nad Saha and Irving Langmuir [128, 71] with reference to a plasma where a single temperature \( T \) may be defined point wise. We first consider the chemical reaction

\[ A^{i+} \rightleftharpoons A^{(i+1)+} + e^{-}, \]  

(3.73)

where \( A^{i+} \) denote a chemical species in its \( i \)-th ionization state (briefly, an \( i \)-ion). At the equilibrium, the densities \( n_e, n_i \) and \( n_{i+1} \) of electrons, \( i \)-ions and \( (i + 1) \)-ions, respectively, satisfy Saha equation

\[ \frac{n_e n_{i+1}}{n_i} = 2 \frac{(2\pi m_e kT)^{3/2}}{\hbar^3} \frac{g_{i+1}}{g_i} \exp \left( -\frac{E_{i+1} - E_i}{kT} \right), \]  

(3.74)

where \( k \) is Boltzmann constant, \( \hbar \) is Planck constant, \( T \) is temperature (assumed unique for all the species), \( m_e \) is the electron mass, \( g_i \) is the degeneracy
of the states of the $i$-ions (i.e., the dimension of the eigenspace associated to the $i$-th eigenvalue of Schroedinger equation for the species at hand) and $E_i$ is the energy required to remove $i$ electrons from a neutral atom, creating an $i$-ion, so that $E_{i+1} - E_i$ is the energy barrier to pass from the $i$-th to the $(i+1)$-th ionization level.

In the case of air plasma, in the typical conditions occurring in low voltage arcs (§3.2.1), one can simply refer to the first ionization reaction

$$A \leftrightarrow A^+ + e^-.$$  \hspace{1cm} (3.75)

The ionization level, that is, the equilibrium between $n_e$ and $n_h$, depends on temperature according to Saha equation, which, in this special case, reads

$$\frac{n_e n_i}{n_n} = 2 \left( \frac{2\pi m_e kT}{h^3} \right)^{3/2} \exp \left( -\frac{E_1}{kT} \right). \hspace{1cm} (3.76)$$

Saha equation (3.74) or (3.76) holds for any given species for which a chemical equilibrium (3.73) or (3.75) holds. For the sake of simplicity we omit references to single species and we deal with an equivalent species, in the spirit of §3.6.1.

In order to determine species densities, we shall use Saha equation together with other two constraints. First, we assume that plasma is neutral point wise, that is, $n_e = Z_i n_i$, with $Z_i \approx 1$, which simply amounts to a local charge conservation principle in absence of charge redistribution. The second constraint is yet another local particle conservation principle in absence of particle redistribution. If one species were not ionized at all (a theoretical case), then $n_n = n_h := n_0$ and $n_i = n_e = 0$. As long as a given species is progressively ionized, the number of ions (and of electrons) increases and that of neutrals decreases, but, since neutrals become ions in a 1:1 ratio, still the sum of heavy particles is conserved and equal to the original $n_0$ value, i.e., $n_i + n_n = n_0$. Of course, one must not look at plasma too close, for otherwise, at the particle micro scale, both the above assumptions cannot hold.

Summing up, one has to solve the simple, second degree algebraic equation

$$\frac{n_e^2}{n_0 - n_e} = 2 \left( \frac{2\pi m_e kT}{h^3} \right)^{3/2} \exp \left( -\frac{E_1}{kT} \right)$$ \hspace{1cm} (3.77)

for $n_e$ and, after that, the densities of other species follows. The result is a set of temperature dependent species and total densities $n_e(T)$, $n_i(T)$, $n_n(T)$ and $n(T) = n_n(T) + n_i(T) + n_e(T)$. We have so far always assumed that the same temperature $T$ may be defined, point wise, for all of the species constituting the air plasma, particularly heavy particles and electrons. This condition describes a thermal equilibrium condition (the so called LTE condition, precisely defined in §3.7.1) which is seen to be the case for most of the arc plasma region.
Nonetheless, there are very particular conditions in a low voltage arc plasma where a two fluid, non-equilibrium model is found to be necessary to correctly describe the observed behavior, namely the arc root spots and the current zero transition, when the arc is close to extinction and which will be addressed by means of black box modeling (see §5.6). In case of a two-fluid model, a couple of possibly different temperatures are defined, particularly $T_e$ for the electron gas and $T_h$ for the heavy particle gas. The problem of generalizing Saha equation to a two-temperature plasma has been long debated, with several different and inequivalent formulations deduced from thermodynamical first principles, together with criticism and corrections (see, e.g., [158, 102, 159, 7, 44]). From the very simple introduction provided in this chapter, the problem is apparently complex due to, among other things, different possible assumptions on ionization mechanisms and the degeneracy of energy eigenstates (which ultimately depend on the gas chemical mixture and on simplifying assumptions of Schroedinger’s equation, when applied to complex molecular structures). Without entering the dispute, we limit ourselves to observing that, in the general non-equilibrium case, species densities become functions of two temperatures, that is, $n_e(T_h, T_e)$, $n_i(T_h, T_e)$, $n_n(T_h, T_e)$ and $n(T_h, T_e) = n_n(T_h, T_e) + n_i(T_h, T_e) + n_e(T_h, T_e)$. In later developments, specifically when developing a black box model for low voltage arcs in §5.6, we shall not need the exact analytic expression of the above functional dependencies, but only the fact that two temperatures appear as independent variables.

We conclude this section with a very rough estimate of the species density in the LTE condition and at typical low voltage arc temperatures. Owing to the great simplifications introduced and to the evident impossibility for any single estimate to hold everywhere in the rapidly changing arc region, we are only concerned with orders of magnitude. The mass density of air at normal conditions is about $\rho_{\text{air}} \approx 1.2 \text{kg/m}^3$. The equivalent molecular mass of fully dissociated but not ionized air is about $m_0 \approx 0.8 m_N + 0.2 m_O \approx 2.4 \cdot 10^{-26} \text{kg}$ (see Table 3.3). Thus the particle density of fully dissociated but not ionized air is about $n_0 = \rho_{\text{air}}/m_0 \approx 5 \cdot 10^{25} \text{m}^{-3}$, where $\rho_{\text{air}}$ has been used for total mass is conserved during dissociation. The first ionization energy for (atomic) nitrogen and (atomic) oxygen is about $E_{1,N} \approx 1402 kJ/mol \approx 2.3 \cdot 10^{-18} \text{J}$ and $E_{1,O} \approx 1314 kJ/mol \approx 2.2 \cdot 10^{-18} \text{J}$, respectively (see Table 3.2). In analogy with the equivalent molecular mass and owing to the similarity of the first ionization energies of two species, we can estimate an equivalent first ionization energy equal to $E_i \approx 2.3 \cdot 10^{-18} \text{J}$. Substituting numerical values into (3.77) one gets Table 3.4.
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\[ T \quad n_e = n_i \quad n_n \quad n \quad \text{[m}^{-3}\text{]} \quad \text{[m}^{-3}\text{]} \quad \text{[m}^{-3}\text{]} \]

<table>
<thead>
<tr>
<th>$T$ [K]</th>
<th>$n_e = n_i$ [$\text{m}^{-3}$]</th>
<th>$n_n$ [$\text{m}^{-3}$]</th>
<th>$n$ [$\text{m}^{-3}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 000</td>
<td>$1.7 \cdot 10^{19}$</td>
<td>$5.0 \cdot 10^{25}$</td>
<td>$5.0 \cdot 10^{25}$</td>
</tr>
<tr>
<td>10 000</td>
<td>$1.2 \cdot 10^{23}$</td>
<td>$5.0 \cdot 10^{25}$</td>
<td>$5.0 \cdot 10^{25}$</td>
</tr>
<tr>
<td>15 000</td>
<td>$2.5 \cdot 10^{24}$</td>
<td>$4.8 \cdot 10^{25}$</td>
<td>$5.3 \cdot 10^{25}$</td>
</tr>
<tr>
<td>20 000</td>
<td>$1.1 \cdot 10^{25}$</td>
<td>$3.9 \cdot 10^{25}$</td>
<td>$6.2 \cdot 10^{25}$</td>
</tr>
</tbody>
</table>

Table 3.4: Ionization equilibrium in low voltage arc plasma according to (single temperature) Saha equation ($n = n_e + n_i + n_n$).

3.6.3 Non Equilibrium Theory

The two-fluid model is useful to describe those phenomena in arc plasma which do not fall under the assumption that particle collisions are frequent enough to allow a proper thermalization. A first fundamental fact to notice, after §3.3.3 and (3.52), is that collisions provide kinetic energy redistribution from fast (i.e., hot) particles to slow (i.e., cold) particles. In other words, thermal equilibrium is produced by means of collisions. Yet, not all collisions have the same effect, for from §3.3.3 we know that a bigger energy transfer is produced when the mass of the colliding particles is similar. Therefore, due to the very unbalanced mass ratio, electron-heavy particle collisions are less effective than electron-electron collisions and heavy particle-heavy particle collisions. As a consequence, electrons and heavy particles as individual species thermalize quicker than the whole plasma as a single fluid. This is the reason why we analyze the first stage of non equilibrium by means of a two-fluid model. It is possible that more extreme non equilibrium conditions require the adoption of more complex models, but the results collected in our study have not shown a particular evidence of the inadequateness of the two-fluid description.

For any species, the particle temperature is related by (3.52) to the mean particle kinetic energy, and thus to the particle mass and velocity, according to gas kinetic theory. Specifically, in the hypothesis of particles with 3 degrees of freedom, one has

\[ \frac{1}{2} m_e \langle v^2 \rangle_e = \frac{3}{2} kT_e \quad (3.78) \]

and

\[ \frac{1}{2} m_h \langle v^2 \rangle_h = \frac{3}{2} kT_h. \quad (3.79) \]

From (3.78) and (3.79), in the LTE condition ($T_e = T_h = T$) one gets $m_e \langle v^2 \rangle_e = m_h \langle v^2 \rangle_h$. Since $m_e \ll m_h$, then

\[ \langle v^2 \rangle_e \gg \langle v^2 \rangle_h \approx \langle v^2 \rangle_i \approx \langle v^2 \rangle_n \quad (3.80) \]
even in the LTE hypothesis. The last two approximate equalities in (3.80) hold due to the kinetic energy redistribution through heavy particle collisions, which are non negligible for the mass similarity argument.

As long as the plasma is hot, a large amount of charge carriers gives rise to a large number of collisions and, therefore, a small mean free path. The mean energy gain from the electric field is thus small if compared with the mean electron kinetic energy and such a gain is quickly and effectively redistributed to ions and neutrals through frequent collisions. As a consequence, electrons, ions and neutrals are thermalized at the same temperature $T$, meaning that one single temperature may be defined point wise in the plasma. In other words, the rate of energy gain from the electric field is comparable to the rate of energy redistribution through collisions and both are small with reference to the mean kinetic energy of particles. This is precisely the main feature characterizing LTE (see §3.7.1), and may be expressed as

$$Ee^{-}\lambda \ll \frac{3}{2}kT,$$

where r.h.s. is the mean kinetic energy (3.52) of the plasma particles (whose definition is sensible because only one temperature field is defined), according to gas kinetic theory and particularly to Boltzmann equipartition theorem.

In presence of suitably strong electric fields (large $E$), i.e., when the discriminating driver in between light and heavy particles cannot be neglected, and/or in presence of suitably long mean free path (large $\lambda$), i.e., when collisions are infrequent, then the action of the electric field on electrons is over a long displacement and thus performs a lot of work before being redistributed (actually slightly redistributed, due to the highly unbalanced mass ratio). Therefore, electron-heavy particle thermalization is prevented, and hypothesis (3.81) is no longer admissible. Non equilibrium theory should be used instead, accounting for electron thermalization and heavy particle thermalization as individual species, and leading to a temperature field $T_e$ for free electrons and a temperature field $T_h$ for heavy particles (ions and neutrals).

Strong electric fields are naturally encountered in high-voltage arc plasma, but also the low voltage realm offers examples of interest. Arc root spots host strong, local electric fields (large $E$) and in a neighborhood of the current zero the plasma is cold and close to extinction and contains relatively few charge carriers (large $\lambda$), so that electron-ion (or heavy particle) thermalization is prevented. In both cases, abandoning the LTE hypothesis appears to be mandatory in order to take an accurate physical picture, so that if one would like to try a computational approach, then either an at least two-fluid model should be adopted, or a suitable technique should be attempted to model what belongs to a finer physical scale that equilibrium theory cannot resolve.
In non equilibrium conditions, inequality (3.80) holds *a fortiori* and is thus even stronger, since for (3.15) free electron velocity is greatly increased by the electric field, contrarily to heavy particles and due to the strongly unbalanced mass ratio. It follows that, in a non equilibrium context, electron temperature exceeds the heavy particle temperature, i.e., $T_e > T_h$, and electrons are hotter than ions and neutrals. On the other hand, $T_h$ is representative of the temperature of most of the matter in the arc plasma, just because $m_e \ll m_h$, and the mass of free electrons may be neglected.

The previous argument has a great impact when we focus on the particle density inside the extinguishing arc. In quasi-neutral plasma, free electron density $n_e$ is equal to positive ion density. Heavy particle density $n_h$ is approximately equal to neutral molecule density, since the amount of charge carriers is usually small in not extremely dissociated plasma (and, *a fortiori*, in cold plasma). For this reason, $T_h$ is the non equilibrium counterpart of $T$, while $T_e$ is a novelty with reference to LTE, introducing electric field dependence, so that $T_e = T_e(E)$. We shall make use of this fundamental theoretical base when developing a black box model for low voltage arcs in §5.6.

### 3.6.4 Debye Shielding

We have so far developed non equilibrium theory with reference to the possibility of a missed thermalization of electrons and ions. There is yet another feature of non equilibrium which has consequences to low voltage arcs, that is, the possibility of a violation of quasi-neutrality.

Plasma is constituted of many moving charges and electrons are actually very fast moving. Therefore the micro scale at the particle level is characterized by a continuous violation of the charge neutrality equilibrium, due to charge fluctuations induced by the random thermal motion of particles. Nonetheless, such a charge unbalance induces restoring electric forces and plasma tends to quasi neutrality, at suitable scale. Furthermore, the temporary charge unbalance cannot give rise to a long ranging difference in the electric potential of the charge distribution, since it is shielded by an opposite charge density gathering all around.

This phenomenon is called *Debye shielding* (or screening) and holds also for charge unbalances induced by other causes. It was first theoretically studied in 1923 by P. Debye and E. Hückel, with reference to charge rearrangements screening electrodes in electrolytes. Noticeably in low voltage circuit breakers, the electrode-plasma interface hosts a gathering of charge density with opposite sign than that of the electrode, thus shielding the latter. Locally, that is, close to the electric contacts, the plasma is not in a quasi neutral condition and at
least a two-fluid model is necessary to describe the physics of such regions. This is yet another reason why a computational approach intended to resolve the scale of arc root spots should not rely on the LTE hypothesis.

We want to determine the length required by plasma to shield a charge unbalance, e.g., induced by an electrode, with reference to an idealized system. The result provides an order of magnitude estimate to be extrapolated in more complex contexts. Let us consider a region of plasma where $n_e \neq n_i$, so that the quasi neutrality assumption is violated. We consider a region of plasma small enough with reference to the scale of temperature variations but, for the sake of generality, we allow $T_e \neq T_i$ (though each one separately is assumed to be constant). We assume electron and ion density distribution be Maxwellian, i.e.,

$$n_s = \overline{n}_s \exp \left( -\frac{u Z_s e^{-}}{k T_s} \right), \quad s \in \{e, i\},$$

where $u$ is the electric potential of the space charge distribution and of external sources and $u Z_s e^{-}$ is the potential energy of the particles of species $s$ immersed into the electric field. If we set $u = 0$ at infinity, then far away from the electrode $\sum_{s \in \{e, i\}} \overline{n}_s Z_s e^{-} = 0$ for global plasma neutrality and $\overline{n}_e = Z_i \overline{n}_i$, since $Z_e = -1$. Assuming that the electric potential is small with reference to the kinetic term $k T_s$, one can consider the linearized expressions

$$n_s \approx \overline{n}_s \left( 1 - \frac{u Z_s e^{-}}{k T_s} \right), \quad s \in \{e, i\}.$$

According to (3.21) and (3.20), the electric charge unbalance is thus

$$\varrho = \varrho_{\text{ext}} + n_i Z_i e^{-} - n_e e^{-} \approx \varrho_{\text{ext}} - \frac{(e^{-})^2 \overline{n}_e (1 + Z_i T_e / T_i)}{k T_e} u$$

where $\varrho_{\text{ext}}$ is the charge located into the elecrode, which is “external” to the plasma. By definition of electric potential, $D = \varepsilon_0 E = -\varepsilon_0 \nabla u$ and according to Gauss law (3.24) the perturbation on the electric potential must satisfy Poisson equation

$$\varepsilon_0 \nabla^2 u = -\varrho \approx -\frac{(e^{-})^2 \overline{n}_e (1 + Z_i T_e / T_i)}{k T_e} u - \varrho_{\text{ext}},$$

which can also be written in operator form as

$$\left( \nabla^2 - \frac{1}{\lambda_D^2} \right) u \approx -\frac{\varrho_{\text{ext}}}{\varepsilon_0},$$

where

$$\lambda_D := \sqrt{\frac{\varepsilon_0 k T_e}{\overline{n}_e (e^{-})^2 (1 + Z_i T_e / T_i)}}$$

(3.82)


is the so-called Debye length.

We followed the definition given in [48], but it must pointed out that in some sources the ion term is neglected, or the whole $1 + Z_i T_e / T_i$ factor is considered but not included into the definition $\sqrt{\varepsilon_0 k T_e / (\bar{n}_e (e^-)^2)}$ of Debye length, or also thermal equilibrium and $Z_i \approx 1$ is assumed, resulting in a factor 2. Actually such possible differences in the definition are meaningless, since we are mainly concerned with orders of magnitude. Substituting numerical values typical of low voltage arc plasma and assuming $T_e = T_i$, so that Saha equation (3.77) may be used, one can estimate the values of the Debye length, as listed in Table 3.5. The Debye length is in the order of nanometers and is thus orders of magnitude smaller than the geometrical size of a breaker or even any part of it.

In order to get an insight on the physical meaning of the Debye length, a simple solution is obtained by abandoning the geometrical complexity of a low voltage circuit breaker and considering the easy-to-solve case in which the electrode is assimilated to a point charge $q$, whose singularity is described by a Dirac delta distribution, i.e., $\varrho_{\text{ext}} = q \delta$. One can always locate the origin of the reference frame where the electrode is located, and, thanks to the spherical symmetry of the problem, the solution is found in the form of a Yukawa potential [39] (or screened Coulomb potential)

$$u(r) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r} \exp \left(-\frac{r}{\lambda_D}\right),$$

where $r$ is the distance from the origin. From the solution, it appears that the Coulomb potential of the perturbing point charge $q$ is shielded on distance scales longer than the Debye length by a shielding cloud of approximate radius $\lambda_D$ of charges of the opposite sign.

Debye length is one of the fundamental parameters used to describe plasma. On spatial scales shorter than Debye length, the microscopic electric fields generated by the charged particles are non negligible and (charged) particles interact by means of the ensuing Coulomb forces. On spatial scales larger

<table>
<thead>
<tr>
<th>$T_e = T_i$ [K]</th>
<th>$\bar{n}_e$ [m$^{-3}$]</th>
<th>$\lambda_D$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>5000</td>
<td>$1.7 \cdot 10^{19}$</td>
<td>$8.3 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>10000</td>
<td>$1.2 \cdot 10^{23}$</td>
<td>$1.4 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>15000</td>
<td>$2.5 \cdot 10^{24}$</td>
<td>$3.8 \cdot 10^{-9}$</td>
</tr>
<tr>
<td>20000</td>
<td>$1.1 \cdot 10^{25}$</td>
<td>$2.0 \cdot 10^{-9}$</td>
</tr>
</tbody>
</table>

Table 3.5: Estimate of the Debye length in low voltage arc plasma.
than Debye length the microscopic fields are screened by the Debye effect and the dynamics of particles is dominated by the smooth, averaged contribution of the whole charge density distribution. Moreover, the extension of the non equilibrium region due to a violation of quasi neutrality is in the order of the Debye length. Implicitly, this amounts to saying that the vast majority of the arc region produced in a low voltage circuit breaker is quasi neutral and described by macro-scale physics. The only possible exceptions must be very limited in size and one may think to model micro-scale singularities and compute macro-scale phenomena, as explained in the next section.

### 3.7 The Magnetohydrodynamic Description

We know from the previous section that there are typical and important conditions in low voltage arc plasma that can only be correctly described by means of non equilibrium theory and (at least) by a two-fluid approach. Nonetheless, provided that arc extinction and possible re-strikes are not addressed and arc root spots are somehow modeled and not resolved, one can try to model and simulate the evolution of an arc at the macro scale by means of a single-fluid approach called magnetohydrodynamics (MHD).

The goal of such a kind of simulation is to reproduce the influence of macro scale features on low voltage circuit breakers, such as the global geometry, exhaust holes, barriers to the propagation of pressure waves, position, number and shape of splitter plates or other ferromagnetic inclusions, the Lorentz force produced by suitably shaped conduction paths, etc. The focus is on the arc global behavior, especially from the fluid dynamic standpoint. Despite the claim to be aprioristic and comprehensive, a critical analysis relying on the assumptions and simplifications outlined so far in the previous sections, as well as those final ones short to come in order to bridge from the two-fluid model to MHD, should clearly clarify that a confrontation with empirical evidence is necessary (and not even simple to achieve).

#### 3.7.1 The Local Thermal Equilibrium (LTE)

MHD is concerned with a plasma in equilibrium, meaning that the rate of temporal and spatial variation of non equilibrium phenomena are below the resolution power of the model. Therefore the magnetohydrodynamic description only makes sense in the framework of the local thermal equilibrium, which we anticipated in §3.6.3 and we now formally introduce. According to Benilov [12], a plasma is said to be in local thermal equilibrium (LTE), sometimes called local thermodynamic equilibrium, when the following hypotheses hold, namely:
1. **Thermal equilibrium:** electron temperature $T_e$ equals (or is very similar to) heavy particle temperature $T_h$, i.e.,

$$T_e = T_h; \quad (3.83)$$

2. **Ionization equilibrium:** electron density $n_e$ equals (or is very similar to) the electron density $n_{\text{Saha}}$ predicted by the Saha equation (3.76) for a single temperature plasma, i.e.,

$$n_e = n_{\text{Saha}}; \quad (3.84)$$

3. **Quasi-neutrality:** the plasma is electrically neutral, both globally and locally (meaning in volumes large enough with reference to the Debye length), i.e.,

$$\rho = \sum_{s \in \{e, i\}} n_s q_s = (Z_i n_i - n_e) e^- = 0,$$

or, equivalently,

$$n_e = Z_i n_i. \quad (3.85)$$

With reference to low voltage arcs, we anticipated in the previous section that the three hypothesis of LTE do not hold in arc root spots, and the hypothesis of thermal equilibrium does not hold when the arc is close to extinction, in the vicinity of current zero. Even though such conditions are obviously very important, with respect to the rest of the arc behavior and having a good, phenomenological model accounting for arc root physics without direct simulation, one can assume that LTE hypothesis is rather well verified.

### 3.7.2 One-Fluid Plasma

The final simplification of plasma equations is obtained by reducing to a one-fluid model. The LTE hypothesis theoretically justifies the adoption of a single temperature field $T$ and a single average velocity field $u$. The procedure is the same already used in §3.6, with some very natural and obvious modifications.

Similarly to (3.65), (3.66) and (3.67), respectively, one defines the **plasma particle density**

$$n := \sum_{s \in S} n_s = n_e + n_h, \quad (3.86)$$

the **plasma mass density**

$$\rho := \sum_{s \in S} \rho_s = \rho_e + \rho_h \quad (3.87)$$

and the **plasma fluid velocity**

$$u := \frac{1}{\rho} \sum_{s \in S} \rho_s u_s = \frac{\rho_e u_e + \rho_h u_h}{\rho_e + \rho_h}. \quad (3.88)$$
From (3.21) and (3.23), respectively, the total charge density
\[ \rho = \sum_{s \in S} \rho_s + \rho_{ext} = \rho_e + \rho_i + \rho_{ext} = \rho_{ext} \] (3.89)
and the total current density
\[ j = \sum_{s \in S} j_s + j_{ext} = j_e + j_i + j_{ext} \] (3.90)
are expressed with reference to electronic, ionic and external contributions. Neutrals obviously do not contribute. By definition, inside the plasma phase the external charge density \( \rho_{ext} \) and the external current density \( j_{ext} \) are identically null, while outside the plasma phase the situation is the opposite with only external contributions present. As regards electric charge density, the quasi-neutrality assumption of LTE hypothesis implies the last equality in (3.89).

Expanding (3.88) and (3.90) inside the plasma phase yields
\[
\begin{align*}
\rho_e m_e u_e + n_h m_h u_h &= (n_e m_e + n_h m_h) u \\
n_e Z_e e^- u_e + n_h Z_h e^- u_i &= j,
\end{align*}
\]
where \( Z_e = -1 \) and \( Z_h \in [0, 1] \). Solving for the velocities of the two fluids, one gets
\[
\begin{align*}
u_e &= u - \frac{1}{n_e} \cdot \frac{1}{m_e/m_h-Z_e/Z_h} \cdot \frac{1}{Z_h e^-} \cdot (j - \rho u) \\
u_h &= u + \frac{1}{n_h} \cdot \frac{m_e/m_h-Z_e/Z_h}{Z_h e^-} \cdot \frac{1}{Z_h e^-} \cdot (j - \rho u),
\end{align*}
\]
where \( \rho = (n_e Z_e + n_h Z_h)e^- = 0 \) for the quasi-neutrality assumption. Recalling (3.10) and (3.12), we can deduce the approximated relations
\[
\begin{align*}
u_e &\approx u - \frac{1}{n_e e^-} \cdot j, \\
u_h &\approx u.
\end{align*}
\] (3.91)
An obvious conclusion is that the average plasma velocity practically coincides with the heavy particle one, due to the strongly unbalanced mass ratio with reference to electrons.

The plasma pressure tensor is defined similarly to (3.68), but with reference to the plasma fluid velocity \( u \), i.e.,
\[
P := \sum_{s \in S} P^m_s \approx P^m_e + P^m_h, \] (3.92)
where
\[ P^m_s := \rho_s \langle (\mathbf{v} - \mathbf{u}) \otimes (\mathbf{v} - \mathbf{u}) \rangle_s, \quad \forall s \in \mathcal{S}. \] (3.93)

Thanks to the second of (3.91), the definition (3.93) practically coincides with (3.69), so that we keep the same symbol for both quantities. For the same reason the final approximated equality in (3.92) holds.

Owing to the previous definitions, one gets the generalized transport equation for mass conservation
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \] (3.94)
for charge conservation
\[ \frac{\partial \varrho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \] (3.95)
for momentum conservation
\[ \frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p - \nabla \cdot \mathbf{\Pi} + \varrho \mathbf{E} + \mathbf{j} \times \mathbf{B}, \] (3.96)
and for energy conservation
\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 + \frac{3}{2} p \right) + \nabla \cdot \left( \frac{1}{2} \rho u^2 \mathbf{u} + \frac{3}{2} p \mathbf{u} + \mathbf{P} \mathbf{u} + \mathbf{q} \right) = \mathbf{j} \cdot \mathbf{E}. \] (3.97)

The quasi neutrality hypothesis \( \varrho = 0 \) allows reducing the charge conservation equation (3.95) to
\[ \nabla \cdot \mathbf{j} = 0, \] (3.98)
according to which the current density field is divergence free. Likewise, the momentum conservation equation (3.96) reduces to
\[ \frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p - \nabla \cdot \mathbf{\Pi} + \mathbf{j} \times \mathbf{B}. \] (3.99)

Equation (3.97) is mathematically inelegant, because it is not in conservative form. This formal nuisance is readily removed by means of Poynting’s theorem [120].

**Theorem 3.1.** (Poynting, 1884)
\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{D} \cdot \mathbf{E} + \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \right) + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{j} \cdot \mathbf{E} \] (3.100)
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Proof. By scalar multiplication of, respectively, Ampere law (3.27) by \(E\) and Faraday law (3.26) by \(H\), then subtracting, one gets

\[
E \cdot \nabla \times H - H \cdot \nabla \times E = E \cdot \frac{\partial D}{\partial t} + H \cdot \frac{\partial B}{\partial t} + E \cdot j.
\]

Applying the product rule for derivatives to divergence to rearrange the l.h.s. and accounting for time independent medium permittivity and permeability in electric and magnetic constitutive equations to handle the r.h.s., (3.100) follows.

Thanks to Poynting’s theorem, energy conservation equation (3.97) may be rewritten as

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 + \frac{3}{2} p + \frac{1}{2} D \cdot E + \frac{1}{2} B \cdot H \right) + \nabla \cdot \left( \frac{1}{2} \rho u^2 \mathbf{u} + \frac{3}{2} p \mathbf{u} + \mathbf{P} \cdot \mathbf{u} + \mathbf{q} + \mathbf{E} \times \mathbf{H} \right) = 0,
\]

which is now fully a conservation law. Particularly, kinetic energy \(1/2 \cdot \rho u^2\) and thermal energy \(3/2 \cdot p\) are now supplemented by electric energy \(1/2 \cdot D \cdot E\) and magnetic energy \(1/2 \cdot B \cdot H\). The divergence term is enriched by the Poynting’s vector \(E \times H\), accounting for electromagnetic energy removed away.

3.7.3 Equation of State

Momentum and energy balance equation contain the total pressure tensor \(\mathbf{P}\). We have already pointed out that, when passing from Boltzmann to fluid equations (single or multi species), the system is not closed and needs being phenomenologically closed. In this section, the spherical part of \(\mathbf{P}\), that is, (hydrostatic) pressure will be related to temperature and density by means of the equation of state. This is the typical approach followed in the case of compressible fluid flows.

The definition (3.51) of temperature is in fact the celebrated equation of state for ideal gasses, which in the case of a single (equivalent) gas reads [40]

\[
p = n k T.
\]

A version more convenient for the form of our equations is

\[
p = \frac{\rho k T}{m},
\]

which has the advantage of only containing those quantities present in mass, momentum and energy equations. To recognize the familiar form of the equation of state, it is sufficient to multiply (3.102) by volume \(V\). Assuming
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Figure 3.7: Mass density of air as a function of temperature and pressure. The ideal gas case (thin lines) is compared with the plasma gas case (thick lines).

that the gas uniformly occupy all the available volume, the quantity $nV$ is the total number of particles in the volume, also equal to the product of the number of moles $N$ by the number of particles in a mole, i.e., the Avogadro constant $N_A \approx 6.022142 \cdot 10^{23} \text{ mol}^{-1}$. Then the (universal) gas constant $R := N_A k \approx 8.314472 \text{ JK}^{-1} \text{ mol}^{-1}$ is defined, so that

$$pV = NRT, \quad (3.104)$$

which is unsuited for CFD but will nonetheless prove to be useful very shortly.

Figure 3.7 shows the mass density of air as a function of temperature and pressure as computed by ABB ab initio code Therminator (thick lines). Any isobaric curve is matched with the relevant isobaric curve as deduced by ideal gas law, i.e., $\rho = mp/kT$ (thin lines), with pressure values indicated on the right part of the graph. The plasma gas deviates from the ideal gas law at high temperatures, as a consequence of the infringements of the hypotheses on which the latter is based. According to kinetic theory, progressively higher temperature activates an increasing number of degrees of freedom and more elaborated models of particles are needed. Also, dissociation increases the number of
particles and therefore increases the pressure. Apparently, higher accuracy is obtained if ideal gas law (3.103) is replaced a more complex relationship (e.g., experimentally sampled).

The theory so far developed references to the very simple point particle model, which is oversimplified for the bi-atomic gasses found in low voltage air plasma. Now we introduce the fundamental concepts that enable us correcting *a posteriori* the equations for the general case. In gas theory, a state function $\mathcal{U}$ is defined, termed the (specific) internal energy, which is usually assumed only to depend on temperature. The term “internal” energy is because it represents the amount of energy stored “inside” the gas and which is increased (respectively, decreased) either by heat insertion (respectively, removal) or by the work done by the external environment against the gas (respectively, by the work done by the gas against the external environment). The *first principle of thermodynamics* states that internal energy is conserved. Precisely,

$$d\mathcal{U} = \delta Q - \delta W,$$

where $d\mathcal{U}$ is an infinitesimal variation of the specific internal energy (obviously positive if it is an increment), $\delta Q$ is a small amount of heat (conventionally positive if inserted into the system) and $\delta W$ is a small work (conventionally positive if the gas does it against the external environment).

Starting from the first principle of thermodynamics, the explicit expression of internal energy may be found by considering the special case of a convenient transformation. Since $\mathcal{U}$ is a state variable, the actual transformation followed is irrelevant and the expression found is universal. As regards the work performed during a gas expansion, $\delta W = p\,dV/\rho V$, where $V$ is the gas volume, $dV$ is its infinitesimal variation and the division by $\rho V$ is to handle specific quantities. So it is convenient to consider an isochoric process (i.e., volume is constant), so that the work vanishes and $d\mathcal{U}$ reduces to $\delta Q$ only. The *specific heat capacity at constant volume* is defined as

$$c_V(T) := \frac{\delta Q}{dT} \bigg|_{V=\text{const}} = \frac{d\mathcal{U}}{dT} \bigg|_{V=\text{const}},$$

where $dT$ is the gas temperature increase relevant to heat insertion $\delta Q$. The first ratio is not a derivative from the mathematical point of view, for heat is not a state variable, and the last equality only holds under the above discussed conditions. Summing up, the internal energy is given by

$$\mathcal{U}(T) = \int_{T_0}^{T} c_V(T') \, dT',$$

where $T_0$ is a reference temperature, such as (but not necessarily) the absolute zero. Internal energy is defined up to an additive constant, but one actually only needs its change, not its value.
3.7. **THE MAGNETOHYDRODYNAMIC DESCRIPTION**

Temperature dependence in $c_V$ has been explicitly indicated so to account for the general case. We have discussed in §3.3.1 that new degrees of freedom are activated at progressively higher temperatures, so that a same amount of energy is spread over a wider range of motion possibilities and the temperature increase is more and more limited. As a consequence, heat capacity is a monotonically non decreasing function of temperature.

If we follow an isobaric (i.e., pressure is constant) gas expansion, then we have an additional $pdV$ term to handle in the first principle of thermodynamics. Differentiating the state equation (3.104), one gets $pdV + VdP = NRDdT$ or, since pressure is constant, $δW = pdV = NRD = \frac{d}{dV} VkdT$. In terms of specific quantities, $δW = \frac{k}{m}dT$, where $W$ has now to be interpreted as a specific work. Since internal energy is a state variable, it only depends on the initial and final temperatures, not on the process followed, and by (3.107) it is equal $δU = c_VdT$. Similarly to (3.106), *specific heat capacity at constant pressure* is defined as

\[
c_p(T) := \frac{δQ}{dT} = \left. \frac{dU + δW}{dT} \right|_{p=\text{const}},
\]

from which *Mayer relation*

\[
c_p = c_V + \frac{k}{m}
\]

follows. Since, at least according to this idealized model, the difference in between specific heat capacity at constant volume and constant pressure is a constant, the same reasonings about temperature dependence hold for $c_p$ as well as for $c_V$.

Similarly to (3.107), *specific enthalpy* is defined as

\[
h(T) := \int_{T_0}^{T} c_p(T')dT'.
\]

Also enthalpy is defined up to an additive constant, and also in this case what is of interest is its change and not its value. By construction,

\[
c_p = \left. \frac{∂h}{∂T} \right|_{p=\text{const}}.
\]

By Mayer relation, $h = U + \frac{k}{m}T$, or also, by the equation of state (3.103),

\[
U = h - \frac{p}{\rho}.
\]

In global (not specific) terms, that is, multiplying by $\rho V$, (3.112) yields the well-known $H = U + pV$ relation, where $H$ and $U$ are the global enthalpy and internal energy, respectively.
The energetic picture of the gas is concluded by the definition of the \textit{(total) specific energy}
\[
\mathcal{E} := \frac{1}{2} u^2 + \mathcal{U} = \frac{1}{2} u^2 + h - \frac{p}{\rho}. \tag{3.113}
\]
The physical interpretation of (3.113) is educational: the total energy of a gas is partitioned exactly according to velocity decomposition\(^3\) (3.41) into a kinetic contribution \(\frac{1}{2} u^2\), accounting for the average, or ordered, kinetic energy and, on the other hand, an internal energy contribution \(\mathcal{U} = \frac{1}{2} \langle v^2 \rangle\), accounting for fluctuating, or disordered, kinetic energy.

When we developed the equations for the kinetic description we used Boltzmann equation for a collection of point shaped particles, which is implicitly equivalent to describing an ideal mono-atomic gas. So we expect to find a legacy of such an assumption inside the single fluid MHD model. This is precisely the case for energy conservation equation (3.97), where from (3.52) we know that in the case of an ideal, mono-atomic gas \(\mathcal{U} = 3 \frac{1}{m} \frac{kT}{2} = \frac{3}{2} \frac{p}{\rho}\). The apparently “strange” \(\frac{3}{2} \rho\) term (everything is multiplied by \(\rho\) in conservation equations) is

\(^3\)Apart from the clearly inessential dismissal of the species subscript.
finally traced back to its physical origin and we see that in equation (3.97) the transported variable is the specific total energy $E$, which justifies the energy conservation name, in the special case of a mono-atomic gas. From (3.106) and (3.111), respectively, we also find the well-known expressions for specific heat capacity at constant volume and constant pressure in the mono-atomic case, that is, $c_V = \frac{3}{2} \frac{k}{m}$ and $c_p = \frac{5}{2} \frac{k}{m}$.

Thanks to the above results, we can now correct the energy equation so that it holds in the general case, and it is thus suited for low voltage arc plasma as well, by simply expressing it in terms of energy as the transported variable, that is,

$$
\frac{\partial}{\partial t} (\rho E) + \nabla \cdot (\rho E u) = j \cdot E - \nabla \cdot (p u) - \nabla \cdot (\Pi \cdot u) - \nabla \cdot q.
$$

(3.114)

The correction is produced by using a more adequate expression for the specific heat capacity instead of the mono-atomic one. Figure 3.8 shows specific enthalpy for air plasma as a function of temperature and pressure and has been computed by ABB ab initio code Therminator. From partial derivation by temperature, the specific heat capacity is obtained as a function of temperature and pressure.

We have already discussed the possibility to use Poynting’s theorem to explicitly reveal the otherwise hidden electromagnetic energy into the total energy count, which is more elegant. Nevertheless the latter formulation is closer to the CFD approach.

### 3.7.4 Fluid Viscosity

The spherical part of $P$, that is, (hydrostatic) pressure, has been related to temperature and density by means of the state equation. As regards the deviatoric part, closure occurs by relating $\Pi$ to viscous forces. In this section we detail how the closure ansatz is deduced in the very simple case of Newtonian fluids.

Viscous forces are present in non ideal fluids (i.e., dissipative, non Eulerian) and account for viscous dissipations due to internal fluid friction. From a microscopic point of view, viscosity is due to the random motion of molecules and ions between neighboring fluid layers. This random motion also transfers some momentum, and therefore it is responsible for diffusing (and homogenizing) the momentum throughout the whole flow. As a matter of fact, the divergence type expression $\nabla \cdot \Pi$, found in momentum balance equation (3.96), represents a momentum outflow from a control volume. Tensor $\Pi$ is also termed shear stress tensor and, in the light of this new interpretation, it collects the force
components acting upon the surfaces of the control volume. From (3.92) and (3.93), it is understood that the total pressure tensor \( P \) is symmetric and so is its deviator \( \Pi \).

We assume the fluid at hand be Newtonian. This is generally accepted (since true to a great extent of precision) for air in normal temperature and pressure conditions. It is important to acknowledge the importance of extending this assumption to plasma as well. In standard fluid mechanics [23], the Newtonian hypothesis is the simplest relationship to define viscous stresses and originates from the observation of laminar mono-directional flows (along direction \( i \)), with shear stress \( \tau \) linearly proportional to the derivative (orthogonally to motion, along direction \( j \)) of the strain rate (or fluid velocity) \( u \), through a proportionality coefficient \( \mu \), or

\[
\tau = -\mu \frac{\partial u_i}{\partial x_j}.
\]

The negative sign is because viscous forces act in between faster and slower moving filaments of fluid, producing a momentum transfer from the faster to the slower ones, and so they act in a direction opposite to that of motion. This also explain why the viscous term is the diffusive term for momentum and, as a matter of fact, why it assumes the diffusive flow form (3.36).

The multi-dimensional counterpart [10] requires a deeper insight into the kinematics of fluid flows but the Newtonian hypothesis still postulates a linear relationship between viscous stresses and velocity gradients. Since viscous stresses are due to differential velocity, we start from observing that two adjacent layers will have, up to higher order infinitesimals, a relative velocity \( \nabla u \).

Nevertheless, the velocity gradient tensor \( \nabla u \) is not symmetric (for, in general, \( \partial_j u_i \neq \partial_i u_j \)) and thus it is incompatible with the symmetry of \( \Pi \), but (as any tensor) it may be decomposed into two terms, the first one symmetric and the other antisymmetric. Precisely, we have \( \nabla u = S + A \), with \( S := \frac{1}{2}(\nabla u + \nabla u^T) \) being symmetric and \( A := \frac{1}{2}(\nabla u - \nabla u^T) \) being antisymmetric. Component wise, this simply means

\[
\partial_j u_i = \frac{1}{2}(\partial_j u_i + \partial_i u_j) + \frac{1}{2}(\partial_j u_i - \partial_i u_j).
\]

The last, antisymmetric contribution \( A \), termed vorticity (or spin) tensor, expresses a rigid rotation of the control volume (with angular velocity \( \omega := \frac{1}{2}\nabla \times u \), \( \chi := 2\omega \) being termed vorticity, as readily understood from its component wise expression) and it is then divergence free, since differential forms obtained from a curl are precisely the kernel of the divergence operator over null-homotopic manifolds; see §A.1.3. Therefore, it does not give rise to
viscous stresses and must not be accounted for when formulating Newtonian viscosity.

So we may only concentrate on the symmetrical part of the velocity gradient tensor which, in its turn, may be decomposed (once again, as any tensor) into a spherical and a deviatoric part. Precisely, we have $S = S_{sph} + S_{dev}$, with $S_{sph} := \frac{1}{3} (\nabla \cdot \mathbf{u}) \mathbf{I}$ being the spherical part and $S_{dev} := \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \frac{1}{3} (\nabla \cdot \mathbf{u}) \mathbf{I}$ being the deviator. By definition, the trace of the spherical part equals the trace of $S$ and thus $S_{sph}$ accounts for volume dilation (rates) and is termed rate of expansion tensor. On the other hand, the deviator is traceless and thus $S_{dev}$ accounts for merely shear strain (rates) and is termed rate of shear tensor.

The most general linear relationship between viscous stresses and strain rates would be of type $\Pi = -3 \mu_b S_{sph} - 2 \mu_d S_{dev}$, $\mu_b$ and $\mu_d$ being fluid properties termed bulk viscosity and dynamic viscosity and expressing the proportionality relationship (coefficients 3 and 2 are only for the sake of convenience). Dimensionally, $[\mu_d] = [\mu_b] = [Ft/\ell^2]$ and in SI units they are both measured in $Pa \cdot s$. Viscosity can also be thought of as the diffusion coefficient for momentum, such as the thermal conductivity is the diffusion coefficient of tem-
perature. A generally universally accepted assumption for mono- or bi-atomic gases is bulk viscosity being negligible, that is, \( \mu_b \approx 0 \). This agrees with our *ab initio* construction of \( \Pi \) as a deviator (and thus a traceless tensor) and we extend this assumption to air plasma simulation\(^4\), so that we only consider the deviatoric part, i.e.,

\[
\Pi = -\mu_d \left( \nabla u + \nabla u^T - \frac{2}{3} (\nabla \cdot u) I \right).
\] (3.115)

When dealing with pure fluid dynamics, in the final formulation of momentum equation (3.96) the mathematical identities \( \nabla \cdot \nabla u = \nabla^2 u \) and \( \nabla \cdot \nabla u^T = \nabla \nabla \cdot u \) are usually exploited to write the contributions of the first two components of the above term. Yet, in the case of magnetohydrodynamics the picture is complicated by viscosity generally being non constant. As a matter of fact, viscosity is affected by temperature and pressure conditions and, consequently, it is position dependent.

Dynamic viscosity of air as a function of temperature and pressure is shown in Figure 3.9, as computed by ABB *ab initio* code Therminator. In normal temperature and pressure conditions, air dynamic viscosity is about \( 1.8 \cdot 10^{-5} \) Pa \( \cdot \) s. Since viscosity is related to internal fluid friction and momentum diffusion, it increases considerably with temperature, as a consequence of increased molecular dissociation and ionization, and can rise one order of magnitude higher. As a consequence, hot plasma tends to stay compact and move throughout the less viscous, surrounding cool air, somehow plowing it. When temperature grows beyond a certain threshold, the hot zone is smeared over a greater extent and viscosity drops down to lower values. Viscosity is also seen to be only weakly pressure dependent.

### 3.7.5 Ohm’s Law

Ohm’s law in a plasma looks rather different from its simpler counterpart in a solid conductor, as stated by the following

**Theorem 3.2.** In a conducting fluid the current density is related to other electrical quantities by the approximated relation

\[
\frac{\partial j}{\partial t} + \nabla \cdot (j \otimes u + u \otimes j - \varrho u \otimes u) = \frac{n_e(e^-)^2}{m_e} (E + u \times B)
\]

\[
-\frac{e^-}{m_e} j \times B + \frac{e^-}{m_e} \nabla \cdot P_e^m - \frac{e^-}{m_e} R_e.
\] (3.116)

\(^4\)Otherwise a different ansatz for \( P \) and how it decomposes would be required.
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Proof. We multiply (3.61), with \( S_s \) neglected, by \( q_s/m_s \) and sum over the species. For the sake of convenience we examine how the balance equation gets modified term-by-term. The storage term is handled in a elementary way as

\[
\sum_{s \in S} \frac{\partial}{\partial t} (n_s q_s u_s) = \frac{\partial}{\partial t} \sum_{s \in S} j_s = \frac{\partial j}{\partial t}.
\]

The convective term requires some long but simple manipulations (most of which have been already used in proving Lemma 3.2), eventually leading to

\[
\sum_{s \in S} \nabla \cdot (n_s q_s (v \otimes v)_s) = \nabla \cdot \sum_{s \in S} \frac{q_s}{m_s} (P^m_s + n_s m_s (u_s \otimes u + u \otimes u_s - u \otimes u)).
\]

The three tensor products at the r.h.s. are simplified straightforwardly. Precisely,

\[
\sum_{s \in S} n_s q_s u_s \otimes u = \sum_{s \in S} j_s \otimes u = j \otimes u \quad \text{and similarly for the transposed term, while} \quad \nabla \cdot \left( q_s m_s \right) (P^m_s + n_s m_s (u_s \otimes u + u \otimes u_s - u \otimes u)).
\]

The two pressure tensor terms, the one relevant to electrons and that relevant to heavy particles, only the former survives, for we apply the usual mass ratio approximation (3.10) and get

\[
\sum_{s \in S} q_s m_s P^m_s = q_e/m_e \left( P^m_e + n_e m_e \left( \frac{q_e}{m_e} \right)^2 \right) \approx -\frac{e^-}{m_e} P^m_e.
\]

A similar argument holds for the collisional term, with \( \mathbf{R}_s \) in place of \( P_s \). The Lorentz term is handled by means of

\[
\sum_{s \in S} \frac{\partial q_s}{m_s} m_s = \sum_{s \in S} n_s q_s^2 \approx \frac{q_e^2}{m_e} \left( n_e + \frac{m_e}{m_h} \left( \frac{q_e}{q_h} \right)^2 n_h \right) \approx \frac{n_e (e^-)^2}{m_e^2}
\]

for the electric portion and

\[
\sum_{s \in S} \frac{\partial q_s}{m_s} u_s \approx n_e (e^-)^2 \frac{u_e}{m_e} \approx \frac{n_e (e^-)^2}{m_e} \left( u - \frac{e^-}{m_e} \right)
\]

for the magnetic portion, where (3.91) has been used. Summing up, (3.116) follows.

Equation (3.116) looks indeed rather impressive. Fortunately, it is possible to introduce a number of simplifications in different practical cases. More details and their justifications, based on order of magnitude arguments, can be found in plasma physics textbooks, such as [48]. Quasi neutrality hypothesis \( \varrho \approx 0 \) removes the last term on the l.h.s. The whole l.h.s. is small with reference to the Lorentz force term (the first on the r.h.s.) and may be therefore neglected. In collisional plasma the \( \partial j/\partial t \) term is small with reference to the resistive term \( (e^-/m_e) \cdot \mathbf{R}_e \) and is thus neglected. Finally, both the so-called Hall term \( (e^-/m_e) \cdot \mathbf{j} \times \mathbf{B} \) and the electron pressure \( (e^-/m_e) \cdot \nabla \cdot P^m_e \) term are small and thus neglected. What is left takes the form

\[
\mathbf{R}_e = n_e e^- \left( \mathbf{E} + \mathbf{u} \times \mathbf{B} \right).
\]

The physical interpretation of (3.117) is simple. Lorentz force on the r.h.s. acts as a driving force on electrons and it is counterbalanced by a frictional force on the l.h.s. With regards to the latter, the resistance to electron motion
is of collisional origin and it is thus directly proportional to mass density and collision frequency $1/\tau_e$, where $\tau_e$ is the electron collision time, i.e., the average time between two consecutive collisions. The resistive term may thus be expressed in the form

$$\mathbf{R}_e = -\rho_e \mathbf{u}_e = -\frac{n_e m_e}{\tau_e} \mathbf{u}_e \approx \frac{m_e}{e \tau_e} \mathbf{j},$$

where $\mathbf{j} \approx -e n_e \mathbf{u}_e$ is current density, here approximated to the electron contribution alone.

Owing to the above reasoning, equation (3.117) takes the familiar form

$$\mathbf{j} = \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

(3.118)

where

$$\sigma = \frac{n_e (e^-)^2 \tau_e}{m_e}$$

(3.119)

is the electrical conductivity of plasma. In solid conductors there is no motion of the supporting medium, i.e., $\mathbf{u} \equiv \mathbf{0}$, and the celebrated $\mathbf{j} = \sigma \mathbf{E}$ form of Ohm’s law is immediately found.

Like virtually all other physical properties, also electrical conductivity is temperature dependent. Actually, it is exactly temperature dependence which constitutes the fundamental feature of arc plasma and decrees the ignition, evolution and extinction or re-strike of arcs in circuit breakers. We consider also the case for pressure dependence, as shown in Figure 3.10, where the electrical conductivity of air plasma is illustrated as a function of pressure and temperature. Data have been produced by ab initio computations with ABB code Therminator. Temperature dependence is exponential, and then strongly non-linear, and is a consequence of gas ionization. Saha equation (§3.6.2) can be used to infer the functional dependence of conductivity on temperature, in the LTE hypothesis, as already agreed for MHD. Air is virtually insulating when temperature is lower than a breakdown barrier approximately corresponding to $6000 \mathbf{K}$, the only charge carriers being the relatively few mobile ions produced by radioactive gases, ultraviolet light, or cosmic rays. A steep ascent is observed next, the so-called avalanche breakdown, since ionization processes are started by an increasing number of free electrons and then they are self sustained. A sort of saturation follows, accompanied by a weaker growth of conductivity, due to the complete ionization of particles, compatibly with the energy levels at hand.

The maximal temperatures attainable in plasma are strongly determined by the radiative properties of the gas, namely absorption coefficient, which, in turn, are determined by energy levels in atomic and molecular orbitals and
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Figure 3.10: Electrical conductivity of air plasma as a function of temperature and pressure, in the range applicable to low voltage arcs.

then by the eigenvalues of Schroedinger equation in quantum mechanics. In case of air plasma, a reasonable temperature range does not exceed $20000 - 30000K$, so that the behavior at higher temperatures is out of scope for the modeling of low voltage air arcs. Pressure dependence is milder and due to the higher number of molecules and atoms which are produced by dissociations and ionizations, resulting in a higher pressure. The electrical conductivity of a hot air plasma is then approximately some $10^4 S/m$, that means, 3 orders of magnitude lower than typical values for metals (e.g., electrical conductivity of copper is $5.9 \cdot 10^7 S/m$, that of aluminium is $1.7 \cdot 10^7 S/m$). When properly cooled, the arc conductivity decreases some 10 orders of magnitude or even more. This is the heart of current interruption and circuit breaking. This is also the backbone of black box modeling.
3.7.6 Magnetic Reynolds Number

The electromagnetic world enters into the fluid world since Lorentz force and Joule heating are present as source terms into momentum and energy conservation equations (3.96) and (3.97), respectively. This creates a mathematical coupling of the equations in the direction from Maxwell to Navier-Stokes equations. On the other hand, the fluid world enters into the electromagnetic world since fluid velocity is present into electric current, as seen either from the original definition (3.22) and (3.23) of the latter or, equivalently, from Ohm’s law in local form (3.118). This creates a mathematical coupling of the equations in the opposite direction, that is, from Navier-Stokes to Maxwell equations.

Due to the full, bidirectional coupling, the whole set of PDE should be solved simultaneously, which amounts to a very cumbersome numerical problem (giving for granted that an analytic solution in the general case of industrial interest is out of reach). Nonetheless, physical intuition suggests that the coupling from Maxwell to Navier-Stokes could be neglected, thus decoupling the problem, provided that the fluid is not moving too fast, since everything originates from the fluid velocity. A simple handling of the basic equations will be more educational. First we introduce the magnetic diffusivity (or magnetic resistivity)

$$\eta := \frac{1}{\mu \sigma}.$$  \hspace{1cm} (3.120)

Then, providing that the flow field is known, the evolution of magnetic flux density may be approximately described by means of one (vector) PDE, termed induction equation of MHD, as stated by the following

**Theorem 3.3.** In the low frequency approximation, the evolution of the magnetic flux density in the plasma phase is approximately ruled by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \hspace{1cm} (3.121)$$

**Proof.** We take the expression of $j$ from Ohm’s law (3.118) and we substitute it into Ampère law (3.27), in the low frequency approximation (3.125). By means of magnetic constitutive equation (3.29) we eliminate $\mathbf{H}$ from the equation obtained thereafter. Then we solve for $\mathbf{E}$, substitute it into Faraday law (3.26) and recall that $\mu$ is position independent in the plasma phase, getting

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}).$$

Great simplifications may be produced on the second term at r.h.s. First, recalling the identity $\nabla \times (fv) = \nabla f \times \mathbf{v} + f \nabla \times \mathbf{v}$, where $f$ is a scalar field and $\mathbf{v}$ is a vector field, one readily obtains

$$\nabla \times (\eta \nabla \times \mathbf{B}) = \nabla \eta \times \nabla \times \mathbf{B} + \eta \nabla \nabla \times \mathbf{B} \approx \eta \nabla \times \nabla \times \mathbf{B},$$

where the approximate equality holds due to the simplifying assumption $\nabla \eta \approx \mathbf{0}$. This is rigorously untrue in arc plasma, because $\sigma$ is a function of temperature and, consequently, of
We shall however neglect such variations and treat $\sigma$ as a constant property. Then, we exploit a second identity in order to cast the double curl operator into a simpler form, that is,
\[ \nabla \times \nabla \times \mathbf{B} = \nabla \nabla \cdot \mathbf{B} - \nabla \cdot \nabla \mathbf{B} = -\nabla^2 \mathbf{B}, \]
where the second equality follows immediately after Gauss law for magnetism (3.25). Thence, (3.121) immediately follows.

Examining the r.h.s. of (3.121), we find a first term involving the fluid motion and then accounting for the flow contribution in displacing charge carriers. For this reason the first term is called inductive or also convective, and stands beside the diffusive, or conductive, term, with the classical Laplacian operator governing diffusive problems (now it should be clear why the denomination of “magnetic diffusivity” has been assigned to $\eta$). This latter contribution is maybe more natural to think of and accounts for magnetic fields not due to the fluid motion (e.g., electrical currents flowing so as the plasma were at rest).

As usual in fluid dynamics, it is natural to compare the two terms by means of a scalar, dimensionless expression (defined point wise) that measures their relative importance. Precisely, the magnetic Reynolds number
\[ Rm := \frac{|\nabla \times (\mathbf{u} \times \mathbf{B})|}{|\eta \nabla^2 \mathbf{B}|} \tag{3.122} \]
is defined, also termed Lundquist number.

If $Rm \ll 1$, then magnetic fields are dominated by the diffusive term and the inductive term may be dropped. This is the realm of the so-called liquid metal MHD and is typical of (relatively) slowly flowing plasmas. Slow here means that the magnetic field lines produced by the electric currents, conceived as assigned inside a frozen plasma, are not deformed by the plasma flow. If this is the case, the magnetic field is weakly coupled to the flow and may be solved for independently, provided that electric currents are known. The magnetic Reynolds number may be also expressed in terms of the characteristic quantities of the physical problem. The numerator of (3.122) scales as $uB/l$ and the denominator scales as $\eta B/l^2$, where $u$, $B$, $l$ and $\eta$ are the typical velocity, magnetic flux density, geometrical length and magnetic diffusivity, respectively, encountered in the problem at hand. Taking the ratio, Lundquist number is seen to scale as
\[ Rm \approx \frac{ul}{\eta} = ul\sigma \mu. \tag{3.123} \]

In the case of low voltage circuit breakers, the gas velocity does not exceed the speed of sound in air and it is thus of the order of some hundreds of $m/s$ ($u \approx 10^2 m/s$). The geometrical scale is of the order of some $cm$ ($l \approx 10^{-2} m$). The electrical conductivity is lower than its asymptotic limit predicted according to
Saha equation \((\sigma \leq 10^4 S/m)\). Magnetic permeability is very similar to that of free space \((\mu \approx \mu_0 = 4\pi \cdot 10^{-7} H/m \approx 10^{-6} H/m)\). Summing up,

\[
Rm \lesssim 10^{-2}.
\] (3.124)

The estimate is confirmed by results of simulations carried out on test cases close to low voltage arc conditions, showing numerical evidence that the magnetic Reynolds number is well below the critical unitary threshold; see §4.4.3.

### 3.7.7 Non-Relativistic Plasma and LF Electromagnetism

The most notable conclusion of the previous section is the possibility to decouple Maxwell equations from Navier-Stokes equations, at least in the direction from the latter to the former. Now we go on considering the problem of the coupling internal to Maxwell equations and we conclude the possibility to decouple the electric field and the magnetic field.

Once again, we need some estimates on the orders of magnitude of the quantities involved. The fundamental fact is that plasma is supposed to move slowly with reference to the speed of light (non-relativistic plasma). Formally, this hypothesis amounts to

\[
u \approx \frac{l}{t} \ll c,
\]

where \(l\) and \(t\) are, respectively, the typical length and time required from the plasma to vary significantly. Since the curl operator consists of spatial derivatives, from Faraday law (3.26) we can approximately infer that \(E/l \approx B/t\) or, equivalently, \(E/B \approx l/t \approx u \ll c\), where \(E\) and \(B\) are the typical values of the electric field and magnetic flux density, respectively. Considering now the Ampère law (3.27) and recalling the electrical and magnetical constitutive relations (3.28) and (3.29), one approximately gets

\[
\frac{\partial \mathbf{D}}{\partial t} \approx \varepsilon_0 \mu_0 l B t \approx \left(\frac{u}{c}\right)^2 \ll 1,
\]

where we made use of (3.31) in the case of vacuum.

Since the rate of variation of the electric flux density is orders of magnitude smaller than the curl of the magnetic field, the latter may be approximately considered insensible to electric field variations in typical arc plasma conditions. This is equivalent to neglect displacement currents with reference to physical currents and mathematically reads

\[
\frac{\partial \mathbf{D}}{\partial t} \approx 0.
\] (3.125)
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Ampère law reduces to $\nabla \times \mathbf{H} = \mathbf{j} = \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B})$. Owing to the low magnetic Reynolds number regime (see §3.7.6), the $\mathbf{u} \times \mathbf{B} \approx \mathbf{0}$ assumption is also justified and the magnetic field may be solved as long as the electric field is known.

The solution of the electric field is simple and decouples from that of the magnetic field if the rate of variation of the magnetic flux density can be neglected in Faraday law (3.26). This assumption is justified because the time constant of the electric network is large compared to that of plasma evolution, so that the rate of variation of electric currents and of the induced magnetic fields are slow. Particularly, we assume

$$\frac{\partial \mathbf{B}}{\partial t} \approx \mathbf{0}, \quad (3.126)$$

so that Gauss law (3.24), Faraday law (3.26) and the constitutive relation (3.28) reduce to the equations of electrostatics

$$\begin{cases} 
\nabla \cdot \mathbf{D} = \rho = 0 \\
\nabla \times \mathbf{E} = 0 \\
\mathbf{D} = \varepsilon \mathbf{E}.
\end{cases} \quad (3.127)$$

In the first of (3.127), $\rho = 0$ due to the quasi-neutrality argument, which holds at the scale resolvable by MHD.

After the electrical problem (3.127) solved, the electrical currents $\mathbf{j}$ are immediately computed by means of Ohm’s law. Gauss law for magnetism (3.25) and Ampère law (3.27) are now decoupled from the electrical problem, resulting into the magnetostatic problem

$$\begin{cases} 
\nabla \cdot \mathbf{B} = 0 \\
\nabla \times \mathbf{H} = \mathbf{j} \\
\mathbf{B} = \mu \mathbf{H}.
\end{cases} \quad (3.128)$$

3.7.8 Material Ablation

The extreme conditions induced by the heat released from the arc are such to produce material ablation from both the metallic parts and the plastic walls of low voltage circuit breakers. The interaction between ablated material, in the form of a vapor, and the arc plasma cannot be neglected [126, 75]. Unfortunately, the problem is rather complex to be described and analyzed in
a precise manner and *ab initio*, and its full understanding is also still incomplete. The little physical information available amounts to some experimental evidence that, if combined to some simple modeling, may provide some qualitative insight into the problem [127]. A controlled material ablation could bring advantages for the arc quenching process, provided that the right materials are chosen. Low voltage circuit breakers manufacturers have been able to use a big wealth of polymers in the last decades, both in the walls surrounding the arc chamber and in gassing material inclusions, purposely located in strategic points inside the arc chamber.

In simple terms, the basic underlying physics is as follows. The heat released from the arc is partially absorbed by solid materials, heating up a portion of their external layers. When their melting and vapor point are reached, then absorbed heat provides the latent heat for the phase transitions. Since the process is endothermic, the arc gets cooler, with a resulting temperature-based decrement of its electrical conductivity. On the other hand, the physical properties of the conducting plasma are modified by addition of new species to the mixture, and it is also possible either that the electrical conductivity be incremented, despite the temperature decrement, or that the overall picture of conducting paths offer new ways to the electric current flow [110]. Particularly, re-deposition of carbonized materials could be a risk of high conductive carbon paths, especially in the case of thermosetting materials used for plastic walls. Also, the pressure increment due to vaporized material affects arc motion [80], possibly in an undesired way. From this description it appears that a first set of parameters to consider includes the melting point and the decomposition temperature [67], that is, the temperature at which the chemical structure of the polymer begin to degrade, as well as the heat capacity [148], vaporization enthalpy [110, 34] (which determine the removed layer thickness), and the electrical conductivity [153] of the vapor.

The arc core reaches temperatures in the order of 10 000 − 20 000 K, but its surrounding zone, being in contact with the polymer, is considerably colder, with temperatures around 3 000 − 5 000 K. This temperature range is of the order of, or sometimes exceeds, some typical dissociation temperatures of polymer components, such as that of C_2H_4 (1 400 K), CH (3 700 K), C_2H (4 500 K) and CO (7 000 K) [139]. More than conduction or convection, heat is transferred from the arc to the polymer by means of radiation. Therefore, the thermal coupling is better if the spectral band of emission from the arc plasma and from the metal contacts and the spectral band of absorption of the polymer coincide or overlap [153].

For Planck’s law, the peak of radiated energy occurs at a frequency which is temperature dependent. Accounting for the typical arc plasma temperatures, the ultraviolet (UV) range has to be considered, and polymers could be engi-
neered so to shift their spectral absorption band at UV frequencies. Despite
the whole picture is not fully understood and the optimal composition is not
well established, some main trends begin to appear. For instance, it seems that
the presence of aromatic rings helps the absorption of UV radiation [59]. Also
the concentration and size effect of fillers in the polymer mixture, related to
UV radiation absorption, has been investigated [53, 97]. The effect of energy
absorbed by radiation due to the different color of the surfaces is also an issue
[127]. The picture is further complicated by ablation-induced morphological
changes in the surface of thermoplastic and thermosetting materials exposed
to electrical arcs, with remarkably different mechanisms in the two cases [84].
As a matter of fact, due to material ablation, a circuit breaker is considerably
modified strike after strike. In the hope to guarantee the same interruption per-
formance in all of the three strikes prescribed by the Standards, an adequate
quantity of gassing material must be provided.

What discussed before is relevant to the potentially beneficial effect of
gassing materials deliberately mounted in circuit breakers. On the other hand,
for the same physical reason, but according to possibly different causes, metal
droplets detach from the metallic conducting path and are cast practically ev-
erywhere inside the arc chamber. This is because of the fluid motion, as can
be seen after examining the circuit breakers after the tests, also the successful
ones.

Owing to the very complex nature of the phenomenon, it is not surprising
that the problem has been mainly approached experimentally, so far. In the
realm of high-voltage circuit breakers, the computational approach has been
attempted with reference to the erosion of the nozzle, usually made of PTFE,
with very approximative and quantitatively unreliable results [47, 169, 24].

Material ablation produces a modification of mass and energy balance equa-
tions. In order to modify the MHD equations accordingly (the multi-fluid level
could be treated analogously), one should start from enthalpy definition (3.110)
and compute the time wise temperature increment in the solid parts made of
gassing materials. When a phase transition temperature is reached, any addi-
tional amount of heat per unit time $Q$ (i.e., a thermal power) is spent to provide
the latent heat, without any increment of the temperature, until the transition
is complete. In particular, when the gassing temperature $T_g$ is reached, then a
mass production rate $\dot{\rho}_g = \dot{Q}/h_g$ is added to the fluid phase, where $h_g$ is the
specific vaporization enthalpy. The most correct way to account for $\dot{\rho}_g$ in mass
balance equation (3.94) is in the form of a BC (boundary condition) for the
fluid sub-domain, relatively to that portion of its boundary neighboring with
the solid phase of gassing materials.

A portion of the energy lent to the solid phase to produce the temperature
increment and the phase transitions is returned to the fluid phase thanks to the ablated material. Precisely, an energy rate $\dot{\rho}_g E_g$ has to be added to the fluid phase, where $E_g$ is the energy content of the ablated material at the temperature at which ablation occurs. Also in this case a BC can be used to account for $\dot{\rho}_g E_g$ in energy balance equation (3.114).

Of course, a very accurate and reliable solution of the temperature and heat flux fields, both inside and on the boundary of solid gassing materials, is a very strong and mandatory pre-requisite, which is probably very difficult both to obtain and to verify. Ablation-induced geometrical modifications of the solid phase should be theoretically accounted for, including melting and the formation of a liquid phase, but a correct modeling of such a difficult phenomenon would require many details from the micro-scale, and particularly from the chemistry of polymers, which are beyond the possibilities of the MHD level of complexity, at least as usually conceived.

### 3.8 Radiation

Both kinetic and fluid equations account for heat transferred by conduction and by convection. Actually, radiation is by far the strongest mechanism for heat transfer in hot plasma, due to the fourth power of temperatures ranging up in between $10,000\,K$ and $20,000\,K$. Needless to say, any physical description of electric arcs must include radiation.

Since radiation is an electromagnetic phenomenon, it can be described by means of the interaction of a gas of photons with the gas of massive particles. We will see in this section how this can be accomplished in the same formal framework developed so far. For the sake of simplicity, we shall work out an additional radiation contribution $q_r$ to the heat term $\nabla \cdot q$ in the energy conservation equation. No other interaction of the photon gas must be included into mass, momentum and energy equations, so that this choice appear to be very simple and convenient.

#### 3.8.1 Radiative Heat Transfer in Participating Media

The simplest way to account for radiation is the so-called Net Emission Coefficient (NEC) method [105], characterized by making the radiated power a suitable (and empirically based) function of local fluid properties, namely temperature and pressure, that is, $\nabla \cdot q_r = f(T, p)$. This means a strictly local model, where any fluid spot radiates energy away, but reabsorption of emitted radiation is totally neglected. This is the limit of the so-called optically thin
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plasma hypothesis. Owing to its great simplicity and extremely reduced computational overhead, the NEC can be used in arc simulations, but, on the other hand, it is generally considered inadequate because in the real context the heat radiated interacts with other fluid spots encountered along the radiation direction, before being dissipated away, outside from the computational domain. This latter situation is referred to as radiation with participating media [99] and claims for a much more complex, non local model, where what happens in some spot is affected by and affects what happens somewhere else.

To better describe the mechanism of radiative heat transfer, the spectral radiative intensity (or spectral radiance) $I_{\nu}(x, \hat{s})$ is introduced, which expresses the intensity of radiation in the point $x \in \mathbb{R}^3$ and in direction $\hat{s} \in \mathbb{S}^2$. The hat over $\hat{s}$ is used to evidence that $\hat{s}$ is a unit vector and then it is bijectively associated to a point over the surface of the bi-dimensional unit sphere $\mathbb{S}^2$. Conceptually, one should think of following a radiation beam and account for the balance of power removed or inserted by means of the interaction with the gas particles found along the way or in the surroundings. Dimensionally, $I_{\nu}$ is a surfacic power flux, per unit solid angle and per unit frequency range. It is thus measured in SI units in $W/(m^2 \cdot Hz \cdot sr)$. The subscript $\nu$ indicates that the quantity is relevant to that prescribed frequency.

The net balance of $I_{\nu}(x, \hat{s})$ along the same propagation direction $\hat{s}$ is ruled by the equation of radiative transfer in a participating medium (or, simply, Radiative Transfer Equation, RTE), that is,

$$\hat{s} \cdot \nabla I_{\nu}(x, \hat{s}) = \alpha_{\nu}(x) I_{bb}^{bb}(x) - (\alpha_{\nu}(x) + \sigma_{\nu}(x)) I_{\nu}(x, \hat{s})$$

$$+ \frac{\sigma_{\nu}(x)}{4\pi} \int_{\mathbb{S}^2} I_{\nu}(x, \hat{s}') \Phi_{\nu}(\hat{s}', \hat{s}) d\hat{s}' .$$

(3.129)

As one may see, time does not appear in this balance equation, because radiation is due to electromagnetic waves traveling at the speed of light, that means, orders of magnitude faster than all of the other physical phenomena involved, so that radiative heat transfer may be assumed to occur virtually instantaneously.

Despite its appearance, the structure of equation (3.129) is rather simple indeed. For the gradient theorem, the l.h.s. is the derivative (and thus the increment) of $I_{\nu}(x, \hat{s})$ in direction $\hat{s}$. The r.h.s. sums up the contributions which produce such a net increment. First we have the emission of the fluid spot in $x$, given by Planck’s law for black body emission (see, e.g., [18] for its deduction)

$$I_{bb}^{bb}(x) = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT(x)} - 1},$$

(3.130)
where $h$ is the Planck constant, $k$ is the Boltzmann constant and $c$ is the speed of light. The black body term is a function of position because temperature $T$ is. The multiplicative function $\alpha_\nu$ is termed absorption coefficient, or absorptivity, and measures both the emitted and the absorbed heat. Then we have $-\alpha_\nu I_\nu$, the absorption of radiation (from the transversed fluid) along its propagation direction, and $-\sigma_\nu I_\nu$, $\sigma_\nu(x)$ being termed scattering coefficient, which measures the scattering of electromagnetic radiation to different directions than $\hat{s}$. Both the two terms are a loss, and then negative. Dimensionally, $[\alpha_\nu] = [\sigma_\nu] = [\ell^{-1}]$ and in SI units they are both measured in $m^{-1}$. Finally, scattering from other directions than $\hat{s}$ is accounted for by integrating over the solid angle the spectral radiative intensity at point $x$ and directed in the general direction $\hat{s}'$. The scattering coefficient $\sigma_\nu$ obviously appears back in the integral term.

The phase function $\Phi_\nu(\hat{s}', \hat{s})$ accounts for the interaction between the two different directions. Usually the assumption is made that the phase function only depend on the angle between the two directions, whose cosine is given by $\hat{s} \cdot \hat{s}'$. In the general case, the phase function is anisotropic and a typical, simple modeling is that of linear anisotropic scattering

$$\Phi_\nu(\hat{s}', \hat{s}) = 1 + C_\nu \hat{s} \cdot \hat{s}'$$

(3.131)

where $C_\nu : \mathbb{R}^3 \to [-1, +1]$ is the linear anisotropic phase function coefficient. When $C_\nu > 0$, then more energy is radiated forward ($\hat{s} \cdot \hat{s}' > 0$ and $\Phi > 1$) than backward ($\hat{s} \cdot \hat{s}' < 0$ and $\Phi < 1$), and vice versa when $C_\nu < 0$. Of course, the case $C_\nu = 0$ is perfect isotropy. Actually, one should not worry too much about anisotropy because, due to the reduced density of the plasma, it can be neglected, that is, $\sigma_\nu \approx 0$, $\forall \nu \in \mathbb{R}^+$, without committing a bigger error than other uncertainties in the breaker’s physics. Therefore, the radiative transport equation gets the considerably simpler form

$$\hat{s} \cdot \nabla I_\nu(x, \hat{s}) = \alpha_\nu(I_{bb}^{\nu} - I_\nu(x, \hat{s}))$$

(3.132)

### 3.8.2 Photohydrodynamics

It is useful to take a step back and look at the radiative transfer equation in a wider context. This will enable us setting radiation into the framework of the kinetic and fluid description of a plasma, as promised. The final result will not only be a unified (and thus mathematically simple and elegant) approach to all phenomena taking place in an electric arc, where all major phenomena in the fluid/plasma phase can be traced back to the interaction between gasses of particles, massive or massless, but also the theoretical development will lead us naturally towards a rather inexpensive computational approach.
If one wants to add photons to the number of particle species, then an additional Boltzmann equation results. But photons are massless particles, always traveling at the speed of light, so that the phase space cannot include velocity, which is no longer a variable. In general, phase space accounts for position and momentum, which is a choice suited both for massive and massless particles. Anyway, in the case of photons we can also equivalently set the problem in a phase space with position $x$, direction of propagation $\hat{s}$ and frequency $\nu$. The number of dimension is still 6, because the three components of $\hat{s}$ are constrained by $|\hat{s}| = 1$. Summing up, if the photon density function $I_{\nu}(x,\hat{s})$ in the phase space $(x,\hat{s},\nu)$ is defined, then Boltzmann equation for photons is obtained (in the usual way as for massive particles) and reads

$$\frac{1}{c} \frac{\partial I_{\nu}}{\partial t} + \hat{s} \cdot \nabla I_{\nu} = \alpha_{\nu} I_{\nu}^{bb} - (\alpha_{\nu} + \sigma_{\nu}) I_{\nu} + \frac{\sigma_{\nu}}{4\pi} \int_{S^2} I_{\nu} \Phi_{\nu}(\hat{s}',\hat{s}) \, d\hat{s}'. \quad (3.133)$$

The time derivative is usually dropped since photons travel with the speed of light, which is considerably higher than the characteristic thermal evolution time for the macroscopic radiative system, so that the steady state condition is assumed, time instant by time instant, for the radiative mechanism of heat transfer. In other words, $I_{\nu}$ is varying slowly in time, if compared to $c$, so that the division by $c$ yields a negligible term.

No term involving the derivative of $I_{\nu}$ with reference to $\hat{s}$ appears in (3.133). For what above, such a term would be equivalent to one with the derivatives of $I_{\nu}$ with reference to momenta, which cannot appear. This is because photons are massless and thus insensible to force interactions.

Collisional contributions are located at the r.h.s. of (3.133), as usual, where the interactions in between the photon gas and other particle gasses are accounted for. Black body emission is an isotropic source term for photons, released by massive particles, whilst absorption is a sink term for photons, absorbed by massive particles. The two terms are the counterpart of ionization and recombination, respectively, for massive particles. Scattering from and into a given direction $\hat{s}$ is another form of interaction with a photon absorbed by a massive particles and released in another direction. This is the counterpart of Coulomb collisions in between massive particles.

We found that (3.129) is exactly Boltzmann equation (3.133) for photons. In other words, we have a kinetic description for radiative heat transfer. Now we take advantage of the fact and use it to derive a fluid description of radiative transfer, or photohydrodynamics, to be used together with magnetohydrodynamics to study the electrical arc plasma in low voltage breakers. We keep the very same approach as in §3.5 and take the first moments of Boltzmann equation to move from the phase space to the physical space.
We start defining the first moments of spectral radiance. The zero-th moment of $I_\nu$, or total incident spectral radiation, reads

$$G_\nu(x) := \int_{S^2} I_\nu(x, \hat{s}) \, d\hat{s}. \quad (3.134)$$

We mention that, integrating over the frequency spectrum and dividing by the speed of light, the radiative energy density per unit volume $\frac{1}{c} \int_{R^+} G_\nu \, d\nu$ is obtained. The first order moment of $I_\nu$, or spectral radiative heat flux, reads

$$q_\nu(x) := \int_{S^2} \hat{s} I_\nu(x, \hat{s}) \, d\hat{s}. \quad (3.135)$$

The second order moment of $I_\nu$ reads

$$P_\nu(x) := \int_{S^2} \hat{s} \otimes \hat{s} I_\nu(x, \hat{s}) \, d\hat{s}. \quad (3.136)$$

Once again, integrating over the frequency spectrum and dividing by the speed of light, the radiative pressure tensor $\frac{1}{c} \int_{R^+} P_\nu \, d\nu$ is obtained.

We need the relevant quantities for black body spectral radiance. The zero-th order moment is defined as

$$G^{bb}_\nu(x) := \int_{S^2} I^{bb}_\nu(x) \, d\hat{s} = I^{bb}_\nu(x) \int_{S^2} \, d\hat{s} = 4\pi I^{bb}_\nu(x), \quad (3.137)$$

where (A.38) has been used. Since black box emission is isotropic and thus independent of direction $\hat{s}$, the first order moment identically vanishes, i.e.,

$$q^{bb}_\nu(x) := \int_{S^2} \hat{s} I^{bb}_\nu(x) \, d\hat{s} = I^{bb}_\nu(x) \int_{S^2} \hat{s} \, d\hat{s} = 0, \quad (3.138)$$

where (A.39) has been used.

If we assume linear anisotropy (3.131), then by means of the previous definitions the collisional scattering term may be simplified. Precisely,

$$\frac{\sigma_\nu}{4\pi} \int_{S^2} I_\nu(x, \hat{s}') \Phi_\nu(\hat{s}', \hat{s}) \, d\hat{s}' = \frac{\sigma_\nu}{4\pi} (G_\nu + C_\nu q_\nu \cdot \hat{s}). \quad (3.139)$$
We can now take the first moments of Boltzmann equation for photons (3.133). The zero-th order moment reads

\[
\int_{S^2} \left( \frac{1}{c} \frac{\partial I_{\nu}}{\partial t} + \hat{s} \cdot \nabla I_{\nu} \right) \, d\hat{s} = \int_{S^2} \left( \alpha_{\nu} I_{\nu}^{bb} - (\alpha_{\nu} + \sigma_{\nu}) I_{\nu} \right) \, d\hat{s}
\]

(3.140)

\[
+ \int_{S^2} \frac{\sigma_{\nu}}{4\pi} \left( G_{\nu} + C_{\nu} \hat{q}_{\nu} \cdot \hat{s} \right) \, d\hat{s}.
\]

The second addendum on the l.h.s. may be handled with greater ease in index notation. Precisely,

\[
\int_{S^2} \sum_{i=1}^{3} \hat{s}_i \frac{\partial I_{\nu}}{\partial x_i} \, d\hat{s} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \int_{S^2} \hat{s}_i I_{\nu} \, d\hat{s}
\]

holds. Accounting for (A.38) and (A.39) and rearranging, the strikingly simple equation

\[
\frac{1}{c} \frac{\partial G_{\nu}}{\partial t} + \nabla \cdot q_{\nu} = \alpha_{\nu} (G_{\nu}^{bb} - G_{\nu})
\]

(3.141)

is obtained, where the scattering terms are no longer present. Similarly, the first order moment of equation (3.133) reads

\[
\int_{S^2} \hat{s} \left( \frac{1}{c} \frac{\partial I_{\nu}}{\partial t} + \hat{s} \cdot \nabla I_{\nu} \right) \, d\hat{s} = \int_{S^2} \hat{s} \left( \alpha_{\nu} I_{\nu}^{bb} - (\alpha_{\nu} + \sigma_{\nu}) I_{\nu} \right) \, d\hat{s}
\]

(3.142)

\[
+ \frac{\sigma_{\nu}}{4\pi} \left( G_{\nu} \int_{S^2} \hat{s} \, d\hat{s} + C_{\nu} \int_{S^2} \hat{s} \otimes \hat{s} \, d\hat{s} \hat{q}_{\nu} \right) .
\]

Also in this case the second addendum on the l.h.s. may be handled with greater ease in index notation. Precisely,

\[
\int_{S^2} \hat{s}_i \sum_{j=1}^{3} \hat{s}_j \frac{\partial I_{\nu}}{\partial x_j} \, d\hat{s} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \int_{S^2} \hat{s}_i \hat{s}_j I_{\nu} \, d\hat{s}
\]

holds. Accounting for (A.39) and (A.40) and rearranging, the once again strikingly simple equation

\[
\frac{1}{c} \frac{\partial q_{\nu}}{\partial t} + \nabla \cdot P_{\nu} = - \left( \alpha_{\nu} + \sigma_{\nu} \left( \frac{1 - C_{\nu}}{3} \right) \right) q_{\nu},
\]

(3.143)
is obtained, where the scattering terms are now in a harmless form. For compactness, we define the diffusion coefficient
\[ \Gamma_{\nu}(x) := \frac{1}{3\alpha_{\nu}(x) + \sigma_{\nu}(x)(3 - C_{\nu}(x))}. \] (3.144)

Finally, neglecting the time derivatives for the reasons already pointed out, we obtain the equations of photohydrodynamics
\[ \begin{align*} \nabla \cdot \mathbf{q}_{\nu} &= \alpha_{\nu}(G_{\nu}^{bb} - G_{\nu}) \\ \nabla \cdot \mathbf{P}_{\nu} &= -\frac{1}{3\Gamma_{\nu}} \mathbf{q}_{\nu}, \end{align*} \] (3.145)
constituting a set of two PDE (one scalar and one in vector form, amounting to four scalar equations overall) per each frequency \( \nu \). Of course, the system cannot be a closed one, for the very same reasons already evidenced in the case of massive particle species when passing from kinetic Boltzmann equation to the general transport equation for fluids. Also, we did not write the second order moment of Boltzmann equation, we stopped one step before.

A possibility to close the system (and definitely not the only physically reasonable one) is the ansatz of a spherical (i.e., isotropic) pressure tensor \( \mathbf{P}_{\nu} = const \cdot \mathbf{I} \). The constant is evaluated by
\[ 3 \cdot const = \text{tr}(\mathbf{P}_{\nu}) = \int_{S^2} |\hat{s}|^2 I_{\nu} \, d\hat{s} = \int_{S^2} I_{\nu} \, d\hat{s} = G_{\nu}. \]
Summing up, we have
\[ \mathbf{P}_{\nu} = \frac{1}{3} G_{\nu} \mathbf{I} \] (3.146)
and now the system is closed. Of course this is only a possibility among many, and a different problem would result from a different closure ansatz, with a mathematically and physically different solution. The goodness of the system closure is only understood by the empirical comparison and the best choice to follow is usually a trade off between accuracy and computational overhead. In the next section the picture will be more complete.

Coming back to our ansatz, the second of (3.145) yields Fick’s law
\[ \mathbf{q}_{\nu} = -\Gamma_{\nu} \nabla G_{\nu}. \] (3.147)
In other words, a scalar potential for radiative heat flux has appeared and, plugging back into the first of (3.145), we reduce to just one Helmholtz PDE per frequency, that is,
\[ \nabla \cdot \Gamma_{\nu} \nabla G_{\nu} = \alpha_{\nu}(G_{\nu} - G_{\nu}^{bb}), \] (3.148)
holding \( \forall x \in \mathbb{R}^3 \) and \( \forall \nu \in \mathbb{R}^+. \) After the PDE solved for \( G_{\nu} \), the spectral radiative heat flux \( \mathbf{q}_{\nu} \) is deduced by (3.147). Finally, in the energy balance
3.8. RADIATION

The overall summation over the frequency spectrum of radiated heat flux is required, that is,

\[ q_r = \int_{R^+} q_\nu \, d\nu. \quad (3.149) \]

One comment is opportune about equation (3.148). The black body term \( \alpha_\nu G_{bb}^\nu \) is the source term, introducing radiation from the air plasma as a fluid into the air plasma as a participating radiative medium. The other terms describe the energy redistribution mechanism, throughout radiation with the participating air plasma medium. Radiatively redistributed heat is then inserted back into the air plasma as a fluid through the divergence of \( q_\nu \), given by (3.149), as a contribution to the divergence of \( \mathbf{q} \) in the energy conservation equation (3.97). This way, part of the heat is transferred from the fluid system to the radiative system, redistributed, and then transferred back from the radiative system to the fluid system.

If the air plasma were not modeled as a participating medium (i.e., without using equation (3.148) or similar redistribution mechanisms) but as a radiatively transparent medium, then the black body radiation would have simply removed energy from the air plasma as a fluid and dissipated it away to the outer environment, with no contribution whatsoever entering back into the fluid system. The absorption coefficient, with its frequency dependence, rules out from the redistribution mechanism those portions of energy associated to frequencies for which the air plasma medium is transparent, or nearly so. The absorption coefficient is also found to be strongly temperature dependent, due to the chemical composition of the gas mixture. Particularly, for a given medium, such as air, or SF\textsubscript{6}, or any other gas, the absorption coefficient peaks at some specified frequency dependent and chemical composition dependent temperature, and then it steeply decreases of many orders of magnitude, practically vanishing for higher temperatures than the peak one. Owing to this behavior, and since the absorption coefficient is also the emission coefficient for a gray matter (i.e., non black), the highest temperature reached by the air plasma in a burning arc is fixed, and coincides with the absorption coefficient peak temperature. If another gas would be used instead of air, the highest temperature would be different, in general, from that of air plasma, but still it would be fixed.

Since light intensity is related to temperature, this theoretical argument explains the experimental evidence that the arc maximal light intensity observed by means of fiber optics is always the same, regardless the current input, and thus regardless the energy input in the form of Joule heating. If a higher current is flowing throughout the arc, so that a higher Joule heating is produced, then the arc is forced by the nature of its radiative properties to enlarge the
size of its cross section. This fact has also been observed by means of fiber optics (see §2.4).

3.8.3 Spherical Harmonics Expansions

Despite the striking simplicity of equation (3.132), the problem of radiation through a Boltzmann equation is numerically very cumbersome, because we must solve a PDE for any direction $\hat{s}$ and for any frequency $\nu$, the latter necessity being due to the strong frequency dependence of the radiative properties of gas-plasma. Some approximated approaches have been elaborated to overcome such problems. The trivial, brute force approach, referred to as the Discrete Ordinate Method (DOM), consists of suitably discretizing both the frequency spectrum and the solid angle. The number of equations resulting therefrom quickly makes the problem size unaffordable, especially in 3D problems such as low voltage circuit breakers. On the other hand, in case of a poor discretization of the solid angle, only a few diffused rays of photons departing from the hot arc zone are studied, corresponding to the discretization directions. Nonetheless, a finely resolved DOM approach may be used as a reference in some simplified cases, in order to compare the results from other, more computationally affordable approaches.

A smarter and lighter approach is the family of so-called $P_N$ methods. Beside the frequency spectrum discretization, angular dependence is resolved by a Fourier series expansion, with spherical harmonics, that is, we assume $I_\nu(\hat{s}) \in L^2(S^2)$ and get

$$ I_\nu(x, \hat{s}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} I_{\nu,lm}(x) Y_{lm}^*(\theta, \phi), $$

(3.150)

where $\theta$ and $\phi$ are spherical coordinates spanning the sphere $S^2$, so that $\hat{s} = \hat{s}(\theta, \phi)$, while $Y_{lm}^*(\theta, \phi)$ constitute an orthonormal base of spherical harmonics (see Appendix §A.3). The coefficients $I_{\nu,lm}(x)$ of the expansion are to be computed by taking the inner products in $L^2(S^2)$ between spectral radiant and the spherical harmonics, i.e., $I_{\nu,lm}(x) = (I_\nu, Y_{lm}^*)_{L^2(S^2)}$. From this point on we proceed in an approximate way, truncating the expansion to

$$ I_\nu(x, \hat{s}) = \sum_{l=0}^{N} \sum_{m=-l}^{l} I_{\nu,lm}(x) Y_{lm}^*(\theta, \phi), $$

(3.151)

which is rigorously inexact and where $N$ is termed the order of the $P_N$ method. The approximation error includes the contributions of the higher order harmonics that have been dropped.
In the Appendix §A.3 it is shown that (3.151) is equivalent to

\[ I_\nu(x, \hat{s}) = \sum_{l=0}^{N} \sum_{i_1=1}^{3} \cdots \sum_{i_l=1}^{3} a^{(l)}_{i_1, \ldots, i_l}(x) \hat{s}_{i_1} \cdots \hat{s}_{i_l}, \tag{3.152} \]

where \( a^{(l)}_{i_1, \ldots, i_l} \) are symmetrical, traceless, rank \( l \) tensors (functionally dependent on the position \( x \in \mathbb{R}^3 \), so it would be more correct to talk about tensor fields). Equivalence holds since, for any given \( l \), the subspace spanned by the \( 2l + 1 \) harmonics \( Y_{l}^{m} \), for \( m \in \{-l, \ldots, +l\} \), coincides with the subspace spanned by the \( l \)-th addendum in (3.152), provided that the above requirements on \( a^{(l)}_{i_1, \ldots, i_l} \) are satisfied. The latter form is more convenient to handle and we use it to deduce the equations to be solved.

For the sake of concreteness we refer to the \( N = 2 \) case, although the method is general and could be applied to any other order. We compute the first moments of (3.152) and after some long but simple computations (all of the necessary integrals may be computed explicitly by means of the results reported in Appendix §A.3) we get

\[ G_\nu(x) = \int_{S^2} I_\nu(x, \hat{s}) \, d\hat{s} = 4\pi a^{(0)}(x), \]

\[ (q_\nu(x))_i = \int_{S^2} \hat{s}_i I_\nu(x, \hat{s}) \, d\hat{s} = \frac{4\pi}{3} a^{(1)}_i(x), \]

\[ (P_\nu(x))_{ij} = \int_{S^2} \hat{s}_i \hat{s}_j I_\nu(x, \hat{s}) \, d\hat{s} = \frac{4\pi}{3} a^{(0)}(x) \delta_{ij} + \frac{8\pi}{15} a^{(2)}_{ij}(x). \]

By construction, in the above formulae no term involving \( l = 3 \) or higher is present. Many contributions proved to be null, due to the orthogonality of the subspaces for different \( l \) and to the properties of tensors \( a^{(l)}_{i_1, \ldots, i_l} \). The above equations may be very easily solved for the tensor coefficients and spectral radiance may be thus expressed in terms of its first moments: Particularly, at the second order,

\[ I_\nu(x, \hat{s}) = \frac{1}{4\pi} \left( G_\nu(x) + 3\hat{s} \cdot q_\nu(x) + \frac{15}{2} \left( P_\nu(x) - \frac{G_\nu(x)}{3} I \right) : (\hat{s} \otimes \hat{s}) \right). \]

In principle one can work with a \( P_N \) method of any given order, substituting spectral radiance in terms of its moments into a suitable number of the first moments of Boltzmann equation for photons (3.133). In practice, great simplicity arises from the choice \( N = 1 \) and the \( P_1 \) method (dipole approximation)
and we will only use it. The adoption of $P_1$ radiation model in low voltage arc simulation is currently a well assessed approach (see, e.g., [146]). Going back to the above computations and retaining only those contributions not higher than first order, one immediately gets

$$I_\nu(x, \hat{s}) = \frac{1}{4\pi} (G_\nu(x) + 3\hat{s} \cdot q_\nu(x)) \quad (3.153)$$

and

$$P_\nu(x) = \frac{G_\nu(x)}{3} I.$$ 

Since we got again the closure ansatz (3.146), we conclude that $P_1$ model is equivalent to the approach outlined in §3.8.2 and the same governing Helmholtz equation (3.148) is obtained. Now the physical meaning of ansatz (3.146) is clear, that is, to consider a spherical spectral radiation pressure tensor is equivalent to adopt a first order approximation of spectral radiance in a spherical harmonics Fourier expansion over the solid angle.

### 3.8.4 Spectral Discretization

After angular dependence discretized by means of spherical harmonics expansions, it is the turn to handle frequency dependence. The radiative properties of matter are strongly frequency dependent. For instance, glass is nearly transparent to visible light and nearly opaque to lower frequency radiated heat. Similarly, air is also much more transparent to visible light than heat, and both air and glass make a good greenhouse.

In the case of the air plasma in low voltage circuit breakers, we can follow the typical approach of the non-gray model and partition the spectrum into a finite number of frequency bands. Radiative properties are assumed to be constant over each band, or, equivalently, the plasma is said to be gray over each band. Formally, frequency bands are defined in the form $B_b := [\nu_{b,l}, \nu_{b,u}]$ or $B_b := (\nu_{b,l}, +\infty)$, with $b \in B$, a suitable index set of cardinality $|B|$. The spectrum is partitioned according to

$$\mathbb{R}^+ = \bigcup_{b \in B} B_b.$$

If the (very natural) choice $\nu_{b+1,l} = \nu_{b,u}$ is performed systemically, then a canonical, non overlapping partition is obtained, even though this is not necessary and portions of the spectrum may be left uncovered. The obvious requirement $B_i \cap B_j = \emptyset$ must anyway be met, so to have a non-overlapping partition. The last approximation corresponds to neglecting the energy conveyed in the uncovered spectrum regions.
We define the sequences
\[ \{\alpha_b\}_{b \in B}, \{\sigma_b\}_{b \in B}, \{C_b\}_{b \in B}, \{\Gamma_b\}_{b \in B}, \{\epsilon_b\}_{b \in B}, \{\rho_b\}_{b \in B}, \]
to gather the assumed constant values of \(\alpha_\nu, \sigma_\nu, C_\nu, \Gamma_\nu, \epsilon_\nu\) and \(\rho_\nu\), respectively, over each band (the last two coefficients will be introduced when handling the boundary conditions). The canonical case amounts to a piecewise constant discretization of radiative properties over the frequency spectrum, so that
\[ \alpha_\nu|_{B_b} = \alpha_b, \sigma_\nu|_{B_b} = \sigma_b, C_\nu|_{B_b} = C_b, \Gamma_\nu|_{B_b} = \Gamma_b, \epsilon_\nu|_{B_b} = \epsilon_b, \rho_\nu|_{B_b} = \rho_b. \]

By integrating over each band, the spectral radiance is substituted by a band radiance
\[ I_b(x, \hat{s}) := \int_{B_b} I_\nu(x, \hat{s}) \, d\nu = \int_{\nu_{b,l}}^{\nu_{b,u}} I_\nu(x, \hat{s}) \, d\nu, \quad \forall b \in B, \quad (3.154) \]
allowing \(\nu_{b,u}\) to be infinite (typically in the case of the last band, to cover the infinite portion of high frequencies). Similarly,
\[ G_b(x) := \int_{B_b} G_\nu(x) \, d\nu, \quad (3.155) \]
\[ q_b(x) := \int_{B_b} q_\nu(x) \, d\nu, \quad (3.156) \]
\[ P_b(x) := \int_{B_b} P_\nu(x) \, d\nu \quad (3.157) \]
are defined \(\forall b \in B\). Since integration over frequency and integration over the solid angle commute, then \(G_b(x), q_b(x)\) and \(P_b(x)\) are the zero-th, first and second order moment, respectively, of \(I_b(x, \hat{s})\), just like \(G_\nu(x), q_\nu(x)\) and \(P_\nu(x)\) are the zero-th, first and second order moment, respectively, of \(I_\nu(x, \hat{s})\).

Finally, black body spectral radiance is replaced by black body band radiance
\[ I_{bb}^b(x) := \int_{B_b} I_{bb}^\nu(x) \, d\nu, \quad \forall b \in B, \quad (3.158) \]
whose zero-th order moment
\[ G_{bb}^b(x) := \int_{B_b} G_{bb}^\nu(x) \, d\nu = 4\pi I_{bb}^b(x), \quad \forall b \in B. \quad (3.159) \]
Unfortunately, the integration of Planck’s law over frequency is not elementary and a primitive function cannot be found in closed form.
The non-gray model is applied verbatim to the $P_1$ model. The band counterpart of (3.147) reads

$$q_b = -\Gamma_b \nabla G_b, \quad b \in B. \tag{3.160}$$

and the infinite sequence of governing Helmholtz PDE (3.148), for $\nu \in \mathbb{R}^+$, is replaced by a finite set of decoupled Helmholtz PDE

$$\nabla \cdot \Gamma_b \nabla G_b = \alpha_b (G_b - G_{bb}^b), \tag{3.161}$$

holding $\forall x \in \mathbb{R}^3$ and $\forall b \in B$. After the PDE solved for $G_b$, the band radiative heat flux $q_b$ is computed and the total radiated heat flux is given by

$$q_r = \sum_{b \in B} q_b.$$

Although it could be inaccurate, the gray model can also be adopted for modeling radiative heat transfer in arc plasma, especially when the numerical solver does not allow finer modeling. The gray model is a special case of the non-gray model when $|B| = 1$, that is, just one band is used and everything is assumed constant over the whole spectrum. One advantage of the gray model is that Planck’s law may be integrated over the whole spectrum in closed form (see (A.44) for details), obtaining

$$I_{bb}^b(x) := \int_0^\infty I_{\nu}^b(x) d\nu = \sigma_{sb} \pi T^4, \tag{3.162}$$

where $\sigma_{sb}$ is the Stefan-Boltzmann constant and from which Stefan-Boltzmann law is deduced after integration over the solid angle.

### 3.8.5 Boundary Conditions

Since radiative heat transfer dominates in electric arc plasma, boundary conditions play a fundamental role in the heat exchange between hot plasma and surrounding cold solid parts, including electrodes and splitter plates. It follows that a correct wall treatment is a necessary condition for a correct modeling of the thermal portion of electron emission from metals into plasma. Moreover, since in low voltage breakers the fluid domain is unbounded and cut such that it is reduced to be bounded, a transmitting boundary condition is also needed for the cutting surfaces. We partition the boundary $\partial \Omega^{rt}$ of the fluid domain $\Omega^{rt}$ into two portions: a reflective portion $\partial \Omega^{rt}_r$ including those parts where the fluid domain is adjacent to some solid domain and a transmitting portion $\partial \Omega^{rt}_t$ including those parts where the fluid domain has been cut. Obviously, $\partial \Omega^{rt} = \partial \Omega^{rt}_r \cup \partial \Omega^{rt}_t$ and the partition is non-overlapping.
We start considering reflective boundary conditions. With reference to Figure 3.11, we consider a spot \( x \in \partial \Omega^rt \), with the fluid domain colored white and the neighboring solid domain colored gray. The boundary needs not being planar, in the general case, but we assume that a unique outward normal unit vector \( \hat{n} \) exists in \( x \). The case of edges may be handled from the theoretical standpoint by applying a local smoothing, involving an arbitrarily small surfacic neighborhood of the edge. From the computational standpoint the problem does not hold since edge points are not used in cell centered discretizations of the computational domain. The normal vector is obviously different point wise over the 2-manifold \( \partial \Omega^rt \) and belongs to the normal bundle \( N\partial \Omega^rt \) of the latter. It is useful to divide the solid angle in the half sphere of outgoing (from the fluid to the solid domain) directions \( S^2_+ := S^2 \cap \{ \hat{s} \mid \hat{n} \cdot \hat{s} > 0 \} \) and the half sphere of incoming (from the solid to the fluid domain) directions \( S^2_- := S^2 \cap \{ \hat{s} \mid \hat{n} \cdot \hat{s} < 0 \} \). The portion \( S^2 \cap \{ \hat{s} \mid \hat{n} \cdot \hat{s} = 0 \} \cong S^1 \) is inessential because it collects directions belonging to the tangent bundle \( T\partial \Omega^rt \) (or, loosely speaking, locally tangent to the boundary).

First, it is useful to remind what happens on the fluid-solid interface. Independently from incident radiation, the solid isotropically emits radiation toward the fluid as a gray body, according to its wall temperature. The spectral radiance emitted is given by Planck’s law (3.130) and reads \( \epsilon_\nu(x)I_{bb}^\nu(x) \), where \( \epsilon_\nu(x) : \partial \Omega^rt \rightarrow [0,1] \) is the emission coefficient, or emissivity. Then we consider what happens when a photon beam hits a solid surface. If \( I_\nu(x,\hat{s}) \) is, as usual, the spectral radiance of the beam for a boundary point \( x \) along an outgoing direction \( \hat{s} \) (big arrow from bottom left), then a portion \( (1 - f_\nu(x))I_\nu(x,\hat{s}) \), with \( f_\nu(x) : \partial \Omega^rt \rightarrow [0,1] \), is mirrored along the unique direction relevant

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**Figure 3.11:** Reflective boundary condition: the fluid domain is colored white and the adjacent solid domain is colored gray.
to $\hat{s}$ by means of Fresnel laws of optics (big arrow to top left). A portion $(1 - \epsilon_\nu(x)) f_\nu(x) I_\nu(x, \hat{s})$ of the remnant is diffusely reflected over many directions (small arrows). Summing up specular and diffuse reflection, an overall quantity $\rho_\nu(x) I_\nu(x, \hat{s})$ is found, where $\rho_\nu(x) : \partial \Omega_{rt}^\nu \to [0, 1]$ is the wall reflectivity. A final portion $\epsilon_\nu(x) f_\nu(x) I_\nu(x, \hat{s})$ is absorbed or transmitted, the latter possibility only occurring in transparent or semitransparent (for the frequency $\nu$ at hand) solids, which is not the case in low voltage circuit breakers.

Now we can finally write the reflective BC as a photon balance in the form

$$\int_{S^2_+} (I_\nu(x, -\hat{s}) - \epsilon_\nu(x) I_{bb}^\nu(x) - \rho_\nu(x) I_\nu(x, \hat{s})) \hat{n} \cdot \hat{s} d\hat{s} = 0,$$

(3.163)

which is known as Marshak boundary condition [114] and holds $\forall x \in \partial \Omega_{rt}$ and $\forall \nu \in \mathbb{R}^+$. The first term under the integral sign is the spectral radiance $I_\nu(x, -\hat{s})$ traveling backward, i.e., from the solid to the fluid along direction $-\hat{s} \in S^2_-$, since in the integral $\hat{s} \in S^2_+$. The sum of such spectral radiance over all backward directions is exactly the sum of isotropic gray body emission $\epsilon_\nu(x) I_{bb}^\nu(x)$ integrated over half space (no matter the direction, due to isotropy) plus the reflected portion of the spectral radiance moving along all forward directions, i.e., from the fluid to the solid along direction $\hat{s} \in S^2_+$. The factor $\hat{n} \cdot \hat{s}$ is the view factor of a boundary portion over the different directions.

In the spirit of photohydrodynamics we want to obtain a BC with no angular dependence. We do this in the case of $P_1$ model. We integrate expression (3.153) over the positive half sphere and, accounting for (A.42) and (A.43), we obtain

$$\int_{S^2_+} I_\nu(x, \pm \hat{s}) \hat{n} \cdot \hat{s} d\hat{s} = \frac{1}{4}(G_\nu \pm 2q_\nu \cdot \hat{n}).$$

(3.164)

Exploiting (3.147), which holds in the case of $P_1$ model, and the gradient theorem to get

$$q_\nu \cdot \hat{n} = -\Gamma_\nu \nabla G_\nu \cdot \hat{n} = -\Gamma_\nu \frac{\partial G_\nu}{\partial n},$$

(3.165)

and plugging back the last two relations into the general Marshak BC (3.163), we get the special case for $P_1$ model, that is,

$$(1 - \rho_\nu)G_\nu + 2\Gamma_\nu(1 + \rho_\nu) \frac{\partial G_\nu}{\partial n} = \epsilon_\nu G_{bb}^\nu.$$ 

(3.166)

We reached a Robin boundary condition (or a linear combination of Dirichlet and Neumann BC) holding $\forall x \in \partial \Omega_{rt}^\nu$ and $\forall \nu \in \mathbb{R}^+$. In [54] a more general frame is studied, leading to an analogous BC. As regards non-gray discretization
of the frequency spectrum, the same reasoning as in §3.8.4 yields

\[(1 - \rho_b)G_b + 2\Gamma_b(1 + \rho_b)\frac{\partial G_b}{\partial n} = \epsilon_b G_{bb}^b,\] (3.167)

still a Robin BC holding \(\forall x \in \partial\Omega^t\) and \(\forall b \in B\). In the very simple, albeit not accurate, gray model, the single Robin BC reads

\[(1 - \rho)G + 2\Gamma(1 + \rho)\frac{\partial G}{\partial n} = 4\epsilon\sigma_{sb}T^4,\] (3.168)

where (3.162) has been recalled.

The case of totally transmitting boundary conditions now follows very simply as a special case of the reflective one. There is no reflection, so that \(\rho_\nu \equiv 0\), and black body emission from the solid wall is now replaced by black body radiation from the outer gas. For the external gas a fixed temperature is assumed and the solid wall emissivity is replaced by the gas absorption coefficient. The transmitting BC is still of Robin type and reads

\[G_\nu + 2\Gamma_\nu \frac{\partial G_\nu}{\partial n} = \alpha_\nu G_{bb}^\nu,\] (3.169)

holding \(\forall x \in \partial\Omega^t\) and \(\forall \nu \in \mathbb{R}^+.\) The band counterpart BC is obtained likewise. It is important to understand that the physical meaning of the transmitting BC is that the radiation going backward into the fluid domain is only the one radiated from the external world. Contrarily to reflective BC, where the solid gets heated (or cooled) by the interaction with the fluid, in the case of transmitting BC the external gas temperature has been fixed, which constitutes an approximation (unavoidable unless a bigger portion of fluid is comprised into the model, and anyway sooner or later occurring). The error involved into the fixed external gas temperature approximation becomes negligible provided that the cutting surface is sufficiently far away from the hot arc plasma region.

### 3.9 Arc Roots

The interface between LV electric arcs and metallic conductors is characterized by remarkable differences with reference to the rest of the arc plasma. Such differences are due to a very different physical behavior at the micro-scale, which can be described by means of non equilibrium theory. The outcome which is perceived at the macro-scale amounts to a localized voltage drop at each arc root spot, whose entity is such that it non negligibly affects the overall electric behavior of a LV circuit breaker. Actually, the current technology employed to produce current limiting circuit breakers and to extinguish LV arcs is exactly based on the exploitation of localized voltage drops at the arc roots.
In what follows next we try to give a qualitative description of the underlying physics, for a better understanding and to serve as a possible starting point for developing a conceptual model which allows to account for the macro-scale behavior avoiding computationally resolving the micro-scale.

### 3.9.1 Qualitative Overview

The physics of arc roots is a complex and still not fully understood subject. As such, it is an active field of research. In what follows next we shall briefly report the findings from high-pressure arcs in high intensity discharge (HID) lamps [12], where conditions similar to low voltage arcs are present. A similar study carried out for low voltage circuit breakers would be very beneficial.

The amplitude of the arc-electrode attachment zone is considerably smaller than the typical amplitude of the arc plasma cloud. This is confirmed by the *a posteriori* analysis of arc burnt electrodes, where the arc “footprints” can be studied and compared with observations of the arc column by optical measurements (see §2.4). A constriction zone is thus identified between the arc column and the electrode surface. This is only the first of a series of perturbations induced by the interaction between the electrode and the arc and manifests itself on different length scales, so that a stratified structure can be defined, with a different physics for each layer (see Table 3.6).

Inside the constriction zone, current density increases close to the electrode, due to the narrowing of the arc cross section, and finally, in the constricted plasma (CP) layer, it becomes comparable to the typical values found at the electrode surface. Heat produced by Joule heating $j \cdot E$ is mainly removed by means of radiation $\nabla \cdot q_r$. Other forms of heat transfer, such as conduction, convection and enthalpy transport by diffusion of species, play a minor role.

Closer to the electrode, the effect of heat removal by means of conduction

| Layer of thermal perturbation | yes | yes | yes | no |
| Ionization layer | yes | yes | no | no |
| Space-charge sheath | yes | no | no | no |
| Electrode | Not Applicable |

Table 3.6: The layered structure of arc roots.
inside the metallic body begin to be non-negligible, and the layer of thermal perturbation (TP) is defined where radiation is no longer the dominating heat transfer mechanism. Up to this level, the LTE hypothesis needs not to be questioned and single-fluid models such as MHD may be used. The LTE hypothesis must be dropped when dealing with the layers even closer to the electrode surface. Looking closer to the electrode, a first layer of thermal non-equilibrium (TN) is present, with electron temperature $T_e$ different from heavy particle temperature $T_h$. Then a ionization layer (IL) is formed, where the electron density $n_e$ differs from that predicted by means of the Saha equation.

Finally, a space charge sheath (SH) is present and the plasma is no longer quasi neutral. The SH can be explained by means of Debye shielding of the electrode and its thickness is thus the Debye length $\lambda_D$. From what in §3.6.4, it is manifest that the fine structure of the arc roots governed by non-equilibrium physics cannot be resolved by a computational grid which, at the same time, is suited to discretize the macro-scale. The cathode and the anode have a space charge sheath of opposite sign. Each electrode with its own space charge sheath behaves like a capacitor. Because the SH is extremely thin, the electric field between the space charge and the electrode surface is strong compared to the field strength outside the layer, in the arc column region, and its direction is practically orthogonal to the electrode surface. In low voltage arcs, the voltage drop across the two SH cannot be neglected with respect to the voltage drop across the remaining part of the arc column. When the arc is split into a series of many segments by ferromagnetic plates, then the number of arc roots is increased, resulting in an increment of the overall arc voltage drop, as already discussed in §1.1.6.

The two electrodes do not behave in a perfectly symmetrical fashion. There are several reasons that explain the breaking of symmetry. First the mass difference between electrons and ions with the consequent difference in their mobility, then the fact that electrons leave the cathode, run through the arc plasma, are then reabsorbed by the anode and are finally available for flowing as current into the connected network, while ions only move from the anode to the cathode. An important role is also played by the work function on the electron emission and re-absorption mechanisms at the electrodes (see the next section).

The strong electric field present in the SH is responsible for accelerating both electrons emitted from the cathode and ions coming from the IL. Electrons are repelled from the electrode, while ions are attracted. It is exactly the kinetic energy gained by electrons from the electric field in the SH that enables them to ionize neutral molecules when in the IL, providing the required ionization energy. Since light and heavy charged particles, i.e., electrons and ions, behave different in an electric field (see §3.6.3), the relevant species exhibit differently
Figure 3.12: Qualitative sketch of the potential energy for an electron along the arc channel and particularly across its roots.

properties and the LTE hypothesis is no longer applicable.

3.9.2 Electron Emission from Metals

The arc attachment region, or the arc root, is characterized by a metal-plasma interface, which has to be trespassed by free electrons in order to form a current. The phenomenon of electron emission from metals is physically described in a different way, according to the particular regime where it occurs, as will be described later [103].

In Figure 3.12 a qualitative sketch is shown of the profile of the potential energy for an electron along the arc channel and across its roots. For the sake of visualization, the drawing is not at scale (neither horizontally, nor vertically), because length and height differ over different orders of magnitude. We remind that the energy $\Delta \mathcal{E}_{A \rightarrow B} = E_B - E_A$ required for a particle with electric charge $q$ immersed into an electric field $E = -\nabla u$ to move from state $A$ (of energy $\mathcal{E}_A$) to state $B$ (of energy $\mathcal{E}_B$) reads

$$\Delta \mathcal{E}_{A \rightarrow B} = - \int_A^B qE \cdot dx = q \int_A^B \nabla u \cdot dx = q \cdot (u_A - u_B) = q \cdot \Delta u_{A \rightarrow B},$$

where the integral is the work performed against the electric field and where the gradient theorem has been exploited to finalize the deduction. The terms $u_A$ and $u_B$ represent the electric potential (i.e., voltage) of state $A$ and $B$, respectively, and $\Delta u_{A \rightarrow B}$ is the relevant voltage drop. In the case of an electron, $q = q_e = Ze^- = -e^-$, so that the potential energy and voltage profile are the same up to a multiplicative constant and have an opposite sign. Both potential energy and voltage are obviously defined up to an additive constant.
The cathode (on the left hand side in Figure 3.12) has a negative charge and emits electrons into the arc channel. The anode (on the right hand side) is positively charged and collects electrons from the arc channel. The arc plasma bridges the two electrodes. The electrical circuit is closed by the rest of the network, so that the anode is connected to the cathode via a generator and loads. Three potential energy profiles are sketched (solid, dash, dash-dotted line), relevant to three different levels of the external electric field, as is later described. Electrons move spontaneously downhill the potential energy curve, that is, in picture from left to right, whereas positive ions move in the opposite way, that is, downhill along the voltage profile (i.e., uphill along the potential energy profile). Electric current has historically been assigned with the latter direction, even though the main charge carriers are electrons, because of their better mobility. Due to the presence of a cloud of particles with opposite sign in front of each electrode (the SH), a large part of the energy decrement is located at the arc roots, so that the arc column itself experiences only a relatively mild slope. This picture is due to the rather mild current dependence of the localized resistances of the arc root spot, and it is characteristic for low voltage circuit breakers. The generator supplies the energy necessary to recover the energy level $E_A$ of state A and to begin a new journey around the circuit.

The basic concept of electron emission is that staying inside a metal conductor is an energetically convenient condition for electrons. A metal conductor is a potential well for electrons, with finitely high walls ($\varepsilon_A < \varepsilon_B$ in Figure 3.12), and, in order to escape from the well, an extra energy $W = \Delta \varepsilon_{A \rightarrow B}$, termed work function, has to be supplied and this amount of energy varies depending on the particular metal material. The particular bond between metal atoms, called metal bond, is such that few electrons per atom are shared and constitute a so-called “sea of electrons” with high internal mobility. The velocity distribution of the gas of shared electrons can be described with kinetic theory and is not uniform. Therefore, a limited number of the most energetic (i.e., fastest) electrons escape from the atomic grid. The rate of electron emission is increased by providing energy to the metal and thus to its electrons, e.g., in the form of heat. The phenomenon is the well-known thermionic emission (or thermionic effect) and is governed by Richardson’s law

$$j = AT^2 \exp \left( -\frac{W}{kT} \right),$$

(3.170)

where $j$ is the magnitude of the current density flowing through the interface, $T$ is the metal surface temperature and $A$ is a material dependent constant.

The presence of an external electric field of intensity $E$ allows lowering the walls of the potential well (dashed profile, to be compared with the continuous
profile in Figure 3.12, and state $B'$ with $\mathcal{E}_{B'} < \mathcal{E}_B$) by an amount
\[ \Delta W = \sqrt{\frac{(e^-)^3 E}{4\pi \varepsilon_0}} \]
and thus it increases the electron emission rate. The phenomenon is called field enhanced thermionic emission, or Schottky emission (or also Schottky effect), and reads
\[ j = A T^2 \exp\left(-\frac{W - \Delta W}{kT}\right), \quad (3.171) \]
at least as long as the external electric field intensity $E$ remains below $10^8 \text{ V/m}$. For higher electric fields (dash-dotted profile in Figure 3.12), quantum tunneling begins to contribute to electron emission in a non negligible way (Fowler-Nordheim effect). Throughout a finite height wall, with higher energy than the eigenstate energy of an electron wave function, the latter experiences an exponential decrement. Therefore a slight reduction of the shape of the wall results in a steep increment of electron emission rate.

When the electric fields are sufficiently strong, the effect of quantum tunneling is dominant over the thermal one and cold field electron emission (CFE), or field emission, occurs. In the intermediate regime in between Schottky emission and CFE, thermo-field emission (TF) occurs, which is described with usually satisfactory accuracy by means of the Murphy-Good equation.

Optical measurements (see §2.4.3) are in agreement with previously reported results [164, 93, 91, 92] and prove the existence of an arc immobility time. A possible explanation is to relate the phenomenon with thermal inertia, that is, the arc root spot may advance only after a suitable surface temperature has been established on the electrode surface. This excludes a CFE mechanism for electron emission. On the other hand, a solely thermal phenomenon would require too high temperatures, regarding the melting point of metals, to justify the observed current density [52]. Thus the relevance of field enhanced electron emission for LV circuit breakers has been postulated. A similar process has also been postulated for HID lamps [12, 13]. The high intensity electric field that is required can been explained either referring to the space charge distribution in front of the electrodes, or with reference to thin oxide layers trapping opposite charged particles [141]. The possibility of a combined action of many different causes, possibly not of equal importance in different phases of arc evolution, has also been postulated.

When electrons reach the anode, they are absorbed and the work function $W$ is released. Obviously, if the two electrode are made of different material, then two different work functions, one per electrode, have to be taken into account. In both the cathode and the anode exists an energy drop, and consequently a
voltage drop. Regarding the anodic voltage drop $\Delta u_{An}$, this is similar to or in the order of the work function $W$. On the contrary, the cathodic voltage drop $\Delta u_{Cat}$, defined as the voltage difference from the inside and the outside of the electrode (i.e., between points $A$ and $C$), could also significantly differ from the work function (see Figure 3.12), and is usually bigger.

### 3.9.3 Heating Mechanisms

From the above, it is apparent that a correct evaluation of the electrode temperature and heating is necessary for a correct modeling of the electron emission in the Schottky regime. A good and reliable model for radiation is only a first step, for other physical phenomena occur and are actually dominant. Most of the results available are relevant to the cathode, since it is recognized to be the most influencing factor on the arc motion.

Radiation gives cooling to the electrode surface. At the same time, the electrode surface is also radiatively heated by the arc plasma cloud. Electrode emission from the cathode is a cooling process since emitted electrons bring away the work function, in the form of kinetic energy. Since the fastest electrons, i.e., those with the highest energy, are emitted, the plasma region close to the cathode is typically hotter than the rest of the plasma. This way of selection is not possible at the anode, since the incoming electrons give up their work function. As a consequence, the anodic plasma region is colder than the cathodic plasma region [41]. Finally, electrode cooling takes also place by conduction into its metallic bulk.

Heavy ions, accelerated in the electric field in the SH, hit the cathode and transfer their kinetic energy to it. Moreover, those ions that reach the cathode recombine with an electron. This results in an amount of energy released to the cathode surface equal to $\mathcal{E}_{\text{ion}} - W$, where $\mathcal{E}_{\text{ion}}$ is the ionization energy and $W$ is, as above, the work function of the emitted electron. The resultant of the above mechanisms is called ionic bombardment. The picture could be complicated by some minor mechanisms. Some fast electrons moving toward the cathode could also overcome the braking and repelling force from the electric field, and yield their kinetic energy to the electrode. Moreover, ions experience collisions in the SH during which part of their kinetic energy is lost.

From the previous discussion it appears that the SH play a key role in determining the physical behavior of the cathodic arc root. This is in contrast to what happens in the anodic arc root. Since electrons have to be inserted into, and not extracted from, the electrode, the current density distribution in the near anode zone is governed by the physics on a wider scale, such as the constriction zone or the region of constricted plasma.
3.10 Conclusions

The current chapter provides a self contained physical description of the arc plasma, with reference to the peculiar conditions found in low voltage circuit breakers. We intended with this to fill a gap in the technical literature about electric arcs, which is usually devoted to the purely electrical aspects. As a matter of fact, while fusion or stellar plasma is a well assessed topic treated by many Authors, industrial applications are still lacking of a rigorous setting.

The review also provides a progressive simplification of the level of detail when modeling plasma, showing the underlying assumptions and limitations. This outlines the scope of and justifies the computational approach based on magnetohydrodynamics; see §4. Particularly, a detailed description is given of those features, such as the arc roots and the current zero, that cannot intrinsically be resolved by the discretization of MHD equations, but need to be modeled with a higher level approach.

In the vast complexity of plasma behavior, we have pointed out the two fundamental mechanisms of particle energy gain from the electric field and particle energy redistribution through collisions. The fulfillment of the local thermal equilibrium is ruled by the balance of the two mechanisms. We have also defined those conditions, namely arc roots and cold plasma close to extinction at current zero, when such equilibrium is reasonably violated. This will provide the natural theoretical framework for the proposed, non equilibrium corrected, low voltage black box model; see §5.

We have provided a description of both flow equations (i.e., the evolution of a gas of massive particles) and radiative heat transfer equations (i.e., the evolution of a gas or massless particles) based on the same mathematical structure. Frequently the two topics are treated separately and independently, and we hope to have recalled the strong relationships in between. This should also explain why we use a fluid solver for both Navier-Stokes and the radiative heat transfer equations in the computational approach; see §4. Finally, we have also been driven by the spirit of formal conciseness and elegance.
Chapter 4

Computational Approach

4.1 Summary

A possible way to implement the magnetohydrodynamic approach in the case of low voltage arcs is introduced in this chapter. First the mathematical structure of the problem is described, consisting in a weakly coupled set of initial boundary value problems (some of which essentially reducing to a sequence of static boundary value problems). The problem is supplemented by the lumped parameter description of the electric network in which the arcing circuit breaker is active. For each problem, the relevant governing equations are first introduced in strong form, which is closer to the physical deduction (see §3.7) and then cast into their weak form, which is closer to the numerical solution.

Then the numerical approximation of the mathematical problem is considered. The spatial discretization of the governing PDE is accomplished by means of the Galerkin method, and particularly the Galerkin-finite volume method for fluid dynamics and radiative transfer, the classical, node-based Galerkin-finite element method for electrostatics and finally the Galerkin-edge finite element method for magnetostatics. The second order BDF scheme, which is implicit and unconditionally stable, is adopted for the time discretization of the set of ODE resulting after the spatial discretization. Only CFD is actually a transient problem, all of the other differential problems reducing to a temporal sequence of steady state problems. A simple scheme is also proposed to resolve the electric network, retaining its integro-differential nature. Data passing from
one differential problem to the other is accomplished by means of a suitable distance weighted interpolation method.

Some preliminary results are shown, in order to illustrate the potential of the method. In general, a reasonable qualitative agreement is observed between simulations and experiments (when available). The \textit{a posteriori} evaluation of the magnetic Reynolds number, carried out in simple conditions which are believed to be close to LV arc plasma, confirms the theoretical estimate of §3.7.6. The main trends in arc dynamics seem to fall within the simulation capabilities of magnetohydrodynamics. Future research has to try to refine the agreement and push it to a quantitative level.

4.2 Problem Setting

In chapter §3 we have outlined the magnetohydrodynamic description of an arc plasma, with particular reference to the conditions that are met for a low voltage circuit breaker. We have already mentioned that the resulting picture is a strong simplification of the real behavior, with some limitations known in advance. Moreover, practically all of the physical properties to be introduced into the equations are hard to be reliably and precisely determined. Finally, the micro scale physics is not fully understood. For this reason and despite the more or less implicit claim, it is hard to believe that a computational approach be truly aprioristic (at least in an initial stage), and a massive experimental validation is required (which is a complex issue on itself).

In this section we propose a possible implementation of the physical concepts into a set of governing equations to be solved. The method follows and is similar to the one introduced by Lindmayer \textit{et al.} \cite{77}. For practical reasons that will shortly become apparent, the implementation, in its turn, introduces additional simplifications both to the governing equations and to the geometry and properties of the object of our study. As a consequence, the mathematical model used for computational purposes is a rather crude approximation of the general physical picture outlined in chapter §3. Nevertheless, a compromise has to be found between the results that can be attained and computational resources that are available. At least in principle, one could think of refining the model to higher levels of realism, but, actually, the current level of complexity to be handled is already challenging.

Finally, for a practical implementation some modeling choices need to be made, fixing some degrees of freedom. As a result, the proposed computational model would necessarily be a special case of the general theoretical framework developed in chapter §3. It is important to realize that other possible imple-
4.2. **PROBLEM SETTING**

Implementations can be conceived for the MHD model. In what follows next we will point out the main modeling choices and discuss the reasons for their opportunity. On the other hand, when possible and opportune, we shall try not to neglect other possible ways and to show how to approach the problem with other well assessed and competing methodologies.

The present and the following sections have a mathematical flavor. Our intention is to contribute to fixing the concepts behind computational magnetohydrodynamics in a clear and solid manner. Very often in engineering practice and in technical literature, computational methods are seen as black boxes, without a clear perception of the inner contents, which is in particular limiting and possibly deceiving, but can also turn out to be dangerous. Also, the weak form (which is in fact solved) remains quite often unmentioned, yielding the illusory impression to the unexperienced reader that the strong form is solved. From the viewpoint of numerical mathematics, the magnetohydrodynamic problem is traditionally approached by means of a collection of consolidated methods, so far with little space for innovative ideas, if any. Our proposed approach is a mixture of aged methods, such as the finite volume method for computational fluid dynamics, and state-of-the-art methods, such as the edge finite element method for computational electromagnetism.

In what follows we will always assume that a global and fixed reference frame has been set, and that \( \mathbf{x} \in \mathbb{R}^3 \) expresses the coordinates of a general point in 3D space according to such a frame. For the sake of concreteness, we shall refer to an orthogonal Cartesian frame, with basis unit vectors \( (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \). Time will be indicated as \( t \), and \( \mathcal{T} := [0, t_f] \), with \( t_f \in \mathbb{R}^+ \), will be a closed time interval over which the arcing phenomenon takes place. The time instant \( t_f \) is the (more or less conventional) ending time of the simulation, to be chosen according to the goal to pursue. The term “air-plasma” will be used to refer to a fluid that can point wise be either gaseous or plasma, depending on local temperature and pressure conditions. Finally, for any given bounded, Lipschitz \([31]\) domain \( \Omega \subset \mathbb{R}^3 \), i.e., a 3-manifold with sufficiently regular boundary; \( \partial \Omega \) is the boundary of \( \Omega \), i.e., a 2-manifold without boundary; \( T\partial \Omega \) is the tangent bundle of \( \partial \Omega \); \( N\partial \Omega \) is the normal bundle of \( \partial \Omega \) and \( \mathbf{n}(\mathbf{x}) \in N\partial \Omega \) is the unit, outward normal vector to \( \partial \Omega \) in \( \mathbf{x} \in \partial \Omega \). The normal derivative, i.e., along direction \( \mathbf{n} \), will be denoted by \( \frac{\partial}{\partial \mathbf{n}} \), which is equal to \( \mathbf{n} \cdot \nabla \).
4.2.1 Formulation of the Fluid Dynamic Problem

Let $\Omega^{fd} \subset \mathbb{R}^3$ be a bounded domain, physically coinciding with the region occupied by the gaseous and plasma phases. Figure 4.1 shows such a domain with reference to an idealized but rather realistic model of MCB, as currently produced by ABB (for a comparison with a real device, see Figure 1.1). The flow and temperature fields of the air-plasma have to be solved in $\Omega^{fd} \subset \mathbb{R}^3$.

Figure 4.1: Domain $\Omega^{fd}$ of the fluid dynamic problem, in the case of an idealized but rather realistic model of MCB from ABB. Splitter plates (solid, dark colored) are not included in the model and the gray face is an exhaust hole to the outer world. The same domain is also used to define the radiative transfer problem.
The governing equations read
\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) &= -\nabla p - \nabla \cdot \Pi + \mathbf{j} \times \mathbf{B} \\
\frac{\partial \rho \mathcal{E}}{\partial t} + \nabla \cdot (\rho \mathcal{E} \mathbf{u}) &= \mathbf{j} \cdot \mathbf{E} - \nabla \cdot (\rho \mathbf{u}) - \nabla \cdot (\Pi \cdot \mathbf{u}) - \nabla \cdot \mathbf{q} \\
p &= \frac{\rho k T}{m} \\
\mathcal{E} &= \frac{1}{2} \mathbf{u}^2 + h - \frac{p}{\rho} \\
h &= \int_{T_0}^{T} c_p(T') dT' \\
\Pi &= -\mu_d (\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3} (\nabla \cdot \mathbf{u}) \mathbf{I}) \\
\mathbf{q} &= -\kappa \nabla T + \sum_{b \in \mathcal{B}} \mathbf{q}_b, \\
\end{align*}
\] (4.1)
to hold \( \forall \mathbf{x} \in \Omega^d \) and \( \forall t \in T \). We recall from chapter \( \S 3 \) the origin and meaning of the variables and equations. As regards the unknowns of the mathematical problem:

- \( \rho(\mathbf{x}, t) \) is the air-plasma mass density;
- \( \mathbf{u}(\mathbf{x}, t) \) is the air-plasma velocity vector;
- \( p(\mathbf{x}, t) \) is the air-plasma pressure;
- \( T(\mathbf{x}, t) \) is the air-plasma temperature;
- \( \mathcal{E}(\mathbf{x}, t) \) is the air-plasma total specific energy;
- \( h(\mathbf{x}, t) \) is the air-plasma specific enthalpy;
- \( \mathbf{q}(\mathbf{x}, t) \) is the total heat flux vector in air-plasma;
- \( \Pi(\mathbf{x}, t) \) is the shear stress tensor in air-plasma.

Some other unknowns appear in problem (4.1), but their determination requires the coupling with other mathematical problems (described in the next sections). Precisely:

- \( \mathbf{j}(\mathbf{x}, t) \) is the electric current density vector;
- \( \mathbf{E}(\mathbf{x}, t) \) is the electric field vector;
- $\mathbf{B}(\mathbf{x}, t)$ is the magnetic flux density vector;
- $\mathbf{q}_b(\mathbf{x}, t)$ is the $b$-th band contribution to the radiative heat flux vector.

In the spirit of weak coupling (see §4.3.1), when solving problem (4.1) the value of such variables has to be assumed known from the solution to other problems. As regards the physical properties of the air-plasma medium appearing in problem (4.1):

- $\mu_d(\mathbf{x}, t)$ is the dynamic viscosity;
- $c_p(\mathbf{x}, t)$ is the specific heat capacity at constant pressure;
- $\kappa(\mathbf{x}, t)$ is the thermal conductivity;
- $m$ is the equivalent molecular mass.

Spatial and time dependence is introduced through temperature and pressure dependence, that is, $\mu_d = \mu_d(T(\mathbf{x}, t), p(\mathbf{x}, t))$ and in analogy with other properties (except for the molecular mass, which is constant). Since physical properties are solution dependent, the problem is nonlinear.

The first three equations are Navier-Stokes equations for a compressible, viscid fluid. They express mass, momentum and energy conservation. The second is actually a vector equation, composed of three terms, i.e., the three components of momentum. Therefore, a total of five PDE appear in (4.1), in the form of general transport equations. The quadratic $\mathbf{u} \otimes \mathbf{u}$ term also makes the problem nonlinear. All other equations are actually the definitions of some quantities as functions of others, and they are not PDE. Yet they are needed match the variables vs. equations count and close the system. The fourth equation is the ideal gas law. It could be replaced by another, non ideal gas law, e.g., an interpolating formula (usually a temperature and pressure dependent mass density) of empirical or theoretical origin. The fifth and sixth equation are the total specific energy and specific enthalpy definitions, respectively. In the latter, $T'$ is a dummy temperature variable and $T_0$ is a reference temperature. The seventh equation is simply the definition of the shear stress tensor, which could also be plugged \textit{sic et simpliciter} in momentum and energy conservation equations. Similarly, the last equation is the definition of the heat flux, constituted by the sum of a conduction component (first addendum on the r.h.s.) and a radiative component.

The differential problem is closed by providing suitable boundary conditions. In the case at hand, the domain boundary $\partial \Omega^d$ is partitioned into a component $\partial \Omega^d_w$, where the sidewalls are located, and a component $\partial \Omega^d_e$, where
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exhaust holes are located; see Figure 4.1, where the exhaust is grayed on the leftmost face and all the rest are walls. The partition covers the whole boundary, i.e., \( \partial \Omega_w^f \cup \partial \Omega_e^f = \partial \Omega^f \), and is non-overlapping, i.e., \( \partial \Omega_w^f \cap \partial \Omega_e^f = \emptyset \), where \( \partial \Omega_w^f := \partial \Omega_w^f \setminus \partial (\partial \Omega_w^f) \) is the interior of \( \partial \Omega_w^f \), and similarly for the other portion of the boundary. On \( \partial \Omega_w^f \times \mathcal{T} \), the fluid cannot have any component normal to the boundary, i.e., \( \mathbf{u} \cdot \mathbf{n} = 0 \), because the fluid cannot cross the rigid walls. Furthermore, a null tangent velocity is also attained on the surface (no-slip wall), so that \( \mathbf{u} = \mathbf{0} \), and a boundary layer is formed. On \( \partial \Omega_e^f \times \mathcal{T} \), a suitable non reflecting boundary condition has to be imposed. Since the air-plasma inside the breaker (where “inside” means in \( \Omega^f \)) is in contact with the external air, and a flow is possible through the exhausts, such a BC has to model an imaginary cut which makes the domain bounded and allows the fluid to flow through, bidirectionally, according to the pressure difference between inside and outside. The exact condition is nonlocal both in space and in time, for a reformulation of flow equations has to be solved over \( \partial \Omega_e^f \times \mathcal{T} [150, 151] \). A series of different approximated but simpler conditions have been also developed in the literature [45], and could probably be helpful.

With the conditions provided, the fluid dynamic problem is an initial boundary value problem.

The above picture is of a fully laminar flow, which is hardly believed to be true for the electric arc problem. From the purely theoretical standpoint, one could always claim to use a direct approach (Direct Navier-Stokes, DNS), that is, to use a fine enough discretization (see §4.3.2) to resolve all of the turbulence scales of practical interest. Actually, such a brute forcing approach seems definitely to be too computationally intensive and thus impractical, at least in the framework of industrial research. On the other hand, the definition, calibration and validation of a suitable turbulence model, e.g., of Reynolds averaged (Reynolds Averaged Navier-Stokes, RANS) or hybrid (Large Eddy Simulation, LES) kind, is way beyond the scope of this work.

The fluid domain is surrounded (and actually shaped) by solid domains, which we have not directly inserted into the mathematical model. As a consequence, \( \Omega^f \) only accounts for fluid domains in our implementation, and holes are left wherever a solid inclusion is located. This is typically the case for the splitter plates and the copper rails (also termed arc runners); see Figure 4.1.

The reason for the choice to exclude solids from the CFD model is that a very fine discretization would be necessary for their accurate description, that means, a very fine computational grid (see §4.3.2) for the numerical approximation of the governing PDE. Rather, one may choose to model the presence of solid walls as semi-infinite half spaces. A thin interface layer may be considered, having a point wise variable temperature. Such a temperature is obtained from the thermal balance in between the heat locally received from the fluid
domain (conductive and convective heat flux from the solution to the fluid dynamic problem, radiative heat flux from the solution to the radiative heat transfer problem) and the heat dispersed into the rest of the semi-infinite solid half space. The latter contribution may be estimated by solving the heat equation over a half space. The net difference between received and dissipated heat results into a temperature increment or decrement. Temperature cut-offs may be used to model phase transitions. During a phase transition, the specific phase transition enthalpy is used to compute the amount of mass undertaking the transition. This approximation is intended for bulky solid inclusions and is immediately seen inadequate for slim ones. Here large or small has to intended with reference to the length covered by heat conduction inside solid materials during the permanence time of the arc close to the fluid-solid interface. A more refined model can be worked out in future research, to describe heat transfer inside bounded domains, such as the splitter plates.

4.2.2 Formulation of the Radiative Transfer Problem

Let \( \Omega^{rt} \subset \mathbb{R}^3 \) be a bounded domain, physically coinciding with the region where the radiative heat transfer with participating media is of interest. The bounded nature of the domain is for the sake of the numerical solution. For practical reasons, it is very convenient - although not strictly necessary - to set \( \Omega^{rt} \equiv \Omega^{fd} \), which we will always assume in this context; see Figure 4.1. The governing equations read

\[
\nabla \cdot (\Gamma_b \nabla G_b) = \alpha_b (G_b - G^{bb}_b), \quad \forall b \in \mathcal{B}.
\]

(4.2)

to hold \( \forall \mathbf{x} \in \Omega^{rt} \) and \( \forall t \in \mathcal{T} \). An equation for each spectral band is defined. Each equation of type (4.2) is a Helmholtz PDE, and the differential operator is the Laplacian. For the \( b \)-th equation, the unknown to solve for is

- \( G_b(\mathbf{x}, t) \), i.e., the total incident spectral band radiation.

After having determined \( G_b \), the \( b \)-th spectral band contribution to the radiative heat flux is immediately obtained as

\[
\mathbf{q}_b = -\Gamma_b \nabla G_b, \quad \forall b \in \mathcal{B}.
\]

(4.3)

Vector \( \mathbf{q}_r(\mathbf{x}, t) = \sum_{b \in \mathcal{B}} \mathbf{q}_b(\mathbf{x}, t) \) is finally passed to the fluid dynamic problem (4.1).

As regards the physical properties of the air-plasma medium appearing in equation (4.2) for the \( b \)-th spectral band:
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• \( \alpha_b(x, t) \) is the spectral band absorption coefficient;
• \( \Gamma_b(x, t) \) is the spectral band diffusion coefficient.

On the other hand,

• \( G_{bb}^b(x, t) \) is the total black body spectral band radiation

and is the forcing term of equation (4.2) for the \( b \)-th spectral band. Spatial and time dependence is introduced through temperature and pressure dependence, that is, \( \alpha_b = \alpha_b(T(x, t), p(x, t)) \) and similarly for \( \Gamma_b \) and \( G_{bb}^b \). Since the radiative properties are made temperature and pressure dependent, and since black body radiation is always temperature dependent, the solution to the problem stemming from (4.2) requires the solution to problem (4.1), which, in its turn, requires the radiative heat flux. In the spirit of weak coupling (see §4.3.1), when solving problem (4.2) the values of temperature and pressure have to be assumed to be known. In this way, problem (4.2) is forcedly made linear at any time it is addressed. The spectral band contribution \( G_{bb}^b \) to black body radiation is considered to be a known function of temperature. Actually, since a non elementary integral is required to produce \( G_{bb}^b \) from \( G_{\nu}^b \), numerical integration can be used, yielding a piece wise linear function of temperature, which can be computed once and for all and conveniently stored in a look-up table for each spectral band.

The differential problem is closed by providing suitable boundary conditions, as detailed in §3.8.5. In the case at hand, \( \partial \Omega_{rt}^r = \partial \Omega_{aw}^d \) and \( \partial \Omega_{rt}^t = \partial \Omega_{ae}^d \). A suitable emissivity and reflectivity has to be chosen for each material and a strong difference exists for metals and plastics.

As already noted in §3.8, time is not directly appearing in (4.2) with its own derivative, because such an equation describes the diffusion of massless particles (the gas of photons) traveling with the speed of light, that is, much faster than the evolution of other phenomena in the electric arc. Therefore the presence of time simply forces the radiative transfer problem to be solved many times, but with a time frequency which is not bounded by a temporal differential constraint. With these conditions and assumptions, the radiative transfer problem becomes a sequence of several boundary value problems.

4.2.3 Formulation of the Electromagnetic Problem

Let \( \Omega^{em} \subset \mathbb{R}^3 \) be a bounded domain, enclosing or coinciding with \( \Omega^{fd} \), i.e., \( \Omega^{em} \supseteq \Omega^{fd} \). Notice that \( \Omega^{fd} \) needs not to be simply connected and thus it...
cannot be considered, in general, a null-homotopic 3-manifold. In the case at hand, the holes left by the splitter plates make $\Omega^{fd}$ to be non simply connected. Still, for the type of geometries involved in practical cases, and usually taking $\Omega^{em}$ as an extension of $\Omega^{fd}$ in a strict sense, i.e., $\Omega^{em} \supset \Omega^{fd}$ (without the equality sign), it is always possible to assume that $\Omega^{em}$ be simply connected. As a consequence, $\Omega^{em}$ may always be considered a null-homotopic 3-manifold. Figure 4.2 shows domain $\Omega^{em}$ with reference to the same idealized but rather realistic model of a MCB, and relevant fluid domain $\Omega^{fd}$, as in Figure 4.1. In the case at hand, domain $\Omega^{em}$ has the form of a truncated cylinder and also contains some metallic bodies, namely the conductive chain (colored gray) and the stack of splitter plates (colored dark gray in the center of the domain). The idealized model can be compared with a real device, as that in Figure 1.1. A large portion of $\Omega^{em}$ is occupied by an outer air box, necessary for the computation of magnetic fields. Figure 4.3 shows a detailed view of the central part of $\Omega^{em}$.

We recall from chapter §3 that the unknown variables of the electromagnetic problem are

- $E(x,t)$, i.e., the electric field vector;
- $B(x,t)$, i.e., the magnetic flux density vector;
- $j(x,t)$, i.e., the electric current density vector.

The electric current density is immediately obtained from the electric field by means of Ohm’s law, i.e.,

$$j = \sigma E. \tag{4.4}$$

When the electromagnetic problem is expressed by means of potentials, as later described, the unknown variables become

- $u(x,t)$, i.e., the electric (scalar) potential;
- $A(x,t)$, i.e., the magnetic vector potential.

The involved physical properties of the media

- $\varepsilon(x,t)$, i.e., the permittivity;
- $\mu(x,t)$, i.e., the permeability;
- $\sigma(x,t)$, i.e., the electric conductivity.
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Figure 4.2: Domain $\Omega^{em}$ of the distributed electromagnetic problem, in the case of an idealized but still realistic model of a MCB from ABB; general view comprehensive of the air box.

In the case of electric conductivity, spatial and time dependence is introduced through temperature and pressure dependence, that is, $\sigma = \sigma(T(x, t), p(x, t))$. As a consequence, the solution to the electromagnetic problem (and particularly the electrostatic one) requires the solution to problem (4.1), which, in its turn, requires the electromagnetic variables. In the spirit of weak coupling (see §4.3.1), when solving the electromagnetic problem the values of temperature and pressure have to be assumed known.

Now we have to consider how to determine electromagnetic quantities. The low frequency approximation allows decoupling Maxwell equations into an electrostatic problem and a magnetostatic one (see §3.7.7). Both problems are defined over $\Omega^{em}$. As regards the electrostatic problem (3.127), the dielectric constitutive equation may be used to remove the electric flux density $D$,
Figure 4.3: Domain $\Omega^{em}$ of the distributed electromagnetic problem, in the case of an idealized but still realistic model of a MCB from ABB; detail view not inclusive of the air box.

yielding

$$\begin{cases} \nabla \cdot \varepsilon \mathbf{E} = 0 \\ \nabla \times \mathbf{E} = \mathbf{0}. \end{cases} \quad (4.5)$$

The second equation of (4.5) states that the electric field is irrotational (i.e., curl free). Since $\Omega^{em}$ is a null-homotopic manifold, theorem A.2 (and the subsequent remark) implies the existence of a scalar electric potential $u$ (i.e., the familiar voltage\(^1\)) whose gradient is (the opposite of) the electric field, or

$$\mathbf{E} = -\nabla u. \quad (4.6)$$

\(^1\)We have used $u$ to refer to the electric potential, because of the long term tradition in electrical engineering. The context will always clarify whether $u$ denotes voltage or $\|\mathbf{u}\|$, i.e., the magnitude of the fluid velocity.
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For the first part of proposition A.1, the identity \( \nabla \times \nabla \equiv 0 \) follows, where 0 is the constant map to zero. Therefore the second of (4.5) is identically satisfied and a familiar (generalized) Laplacian problem remains to be solved, namely

\[
\nabla \cdot \varepsilon \nabla u = 0,
\]

(4.7)

to hold \( \forall x \in \Omega^{em} \) and \( \forall t \in \mathcal{T} \). The electrostatic problem is linear. Since constants are in the nullspace of the gradient, voltage \( u \) is defined up to a constant.

There are different ways to supplement (4.7) with BC. We choose to partition the boundary \( \partial \Omega^{em} \) into three components, namely \( \partial \Omega^u \), where a fixed electric potential is imposed, \( \partial \Omega^j \), where a prescribed electric current density is imposed, and finally \( \partial \Omega^i \), where a null electric current density is imposed. The portions \( \partial \Omega^u \) and \( \partial \Omega^j \) correspond to the two electric terminals of the circuit breaker (a vertical, rectangular facet on the front and one on the rear of Figure 4.3, where the conducting path is terminated; which of the two is \( \partial \Omega^u \) and which is \( \partial \Omega^j \) is irrelevant from the theoretical point of view), whilst \( \partial \Omega^i \) represents the rest of the computational domain boundary, which is supposed to be insulating. The partition covers the whole boundary, i.e., \( \partial \Omega^u \cup \partial \Omega^j \cup \partial \Omega^i = \partial \Omega^{em} \), and is non-overlapping, i.e., \( \partial \Omega^u \cap \partial \Omega^j = \partial \Omega^j \cap \partial \Omega^i = \partial \Omega^i \cap \partial \Omega^u = \emptyset \), where \( \partial \Omega^u := \partial \Omega^u \setminus \partial(\partial \Omega^u) \) is the interior of \( \partial \Omega^u \), and similarly for the other portions of the boundary. On \( \partial \Omega^u \times \mathcal{T} \), the homogeneous Dirichlet boundary condition

\[
u = 0
\]

(4.8)
is imposed. The choice of zero as reference potential is merely conventional and irrelevant, since \( u \) is defined up to a constant. This is also the trivial way by which the electrostatic problem in terms of its potential is given a unique solution, for the nullspace of its operator is eliminated. On \( \partial \Omega^j \times \mathcal{T} \), the generally inhomogeneous Neumann boundary condition

\[
-\sigma \frac{\partial u}{\partial n} = j
\]

(4.9)
is imposed. The condition follows from (4.4) and (4.6). With \( j \) we intend the normal component \( n \cdot j \) of vector \( j \), which we actually may assume to be normal to \( \partial \Omega^j \), without loss of generality. At any time instant \( t \), the electric current density must add up to the total electric current \( i(t) \) flowing into the breaker, that is,

\[
\int_{\partial \Omega^j} j(x, t) \, d\Gamma = i(t), \quad \forall t \in \mathcal{T},
\]

(4.10)
where $dT$ is the infinitesimal element of the boundary surface. On $\partial \Omega_{i}^{cm} \times \mathcal{T}$, the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0$$

(4.11)

is imposed, which can also be considered a special case of the previous condition.

As regards the magnetostatic problem (3.128), the magnetic constitutive equation may be used to remove the magnetic field $\mathbf{H}$, yielding

$$\begin{cases}
\nabla \cdot \mathbf{B} = 0 \\
\nabla \times \frac{1}{\mu} \mathbf{B} = \mathbf{j}.
\end{cases}$$

(4.12)

The first of (4.12) states that the magnetic flux density is solenoidal (i.e., divergence free). Since $\Omega^{cm}$ is a null-homotopic manifold, theorem A.3 (and the subsequent remark) implies the existence of a magnetic vector potential $\mathbf{A}$ whose curl is the magnetic flux density, or

$$\mathbf{B} = \nabla \times \mathbf{A}.$$  

(4.13)

For the second part of proposition A.1, the identity $\nabla \cdot \nabla \times \equiv 0$ follows, where 0 is the constant map to zero. Therefore the first of (4.12) is identically satisfied and a curl-curl problem remains to be solved, namely

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{A} = \mathbf{j},$$

(4.14)

to hold $\forall \mathbf{x} \in \Omega^{cm}$ and $\forall \mathbf{t} \in \mathcal{T}$. The presence of ferromagnetic inclusions (i.e., the splitter plates; see the remark in §3.4.3) makes nonlinear the otherwise linear magnetostatic problem.

For theorem A.3, the magnetic vector potential $\mathbf{A}$ is defined up to a gradient. A common choice to fix the arbitrary gradient and eliminate the nullspace of the curl-curl operator is to assume that the vector potential be divergence free, i.e.,

$$\nabla \cdot \mathbf{A} = 0.$$  

For understanding the latter condition, called Coulomb gauge, one may choose an arbitrary (and assumed known) vector potential $\mathbf{A}_{0}$, which need not satisfy the gauge, and express any other vector potential as $\mathbf{A} = \mathbf{A}_{0} + \nabla \psi$, where $\nabla \psi$ is an arbitrary gradient. If $\mathbf{A}$ has to satisfy the Coulomb gauge, then condition $-\nabla^{2} \psi = \nabla \cdot \mathbf{A}_{0}$ is imposed on the scalar field $\psi$. If suitable boundary conditions are provided, such as Dirichlet ones over the whole boundary of the domain at hand, then a well posed Poisson boundary value problem is obtained, admitting
a unique solution $\psi$ and thus fixing $A$ in a unique way. The Coulomb gauge is difficult to be imposed numerically, and other techniques are preferred.

PDE (4.14) must be supplemented by boundary conditions. Typical electromagnetic BC impose that a vector field be tangent or rather normal to the boundary. To formalize the two cases, we need a preliminary remark. Let $v \in \mathbb{R}^3$ be any given vector. The tangent condition is straightforwardly imposed by means of a scalar product, because $n \cdot v = 0$ means that $v$ is orthogonal to $n$ and thus lies in the tangent plane to the boundary surface. To get to the normal condition, we note that $v$ may be decomposed with respect to the normal $n$ as $v = v_n + v_t$, with $v_n \in N\partial\Omega^{em}$ being the normal component and $v_t \in T\partial\Omega^{em}$ being the tangent component. The latter is uniquely defined after $v$ and $v_n$. By definition $v_n := v \cdot n$. Therefore, $n \times v_n = n \times v \cdot n \equiv 0$, because a vector product is always orthogonal to its factors. As a consequence, $n \times v = n \times (v_n + v_t) = n \times v_t$. It follows that the vector product condition $n \times v = 0$ implies $n \times v_t = 0$ and, since $v_t$ is orthogonal to $n$ and thus it cannot be parallel to the latter, then the condition is equivalent to $v_t = 0$ and $v = v_n$. In other terms, $n \times v = 0$ implies that $v$ be parallel to $n$, or $v \in N\partial\Omega^{em}$.

After the above remark and similarly to electrostatics, another partition of the boundary $\partial\Omega^{em}$ is introduced for magnetostatics, consisting of the two components $\partial\Omega^{em}_h$ and $\partial\Omega^{em}_b$. As usual, also this partition covers the whole boundary, i.e., $\partial\Omega^{em}_h \cup \partial\Omega^{em}_b = \partial\Omega^{em}$, and is non-overlapping, i.e., $\partial\Omega^{em}_h \cap \partial\Omega^{em}_b = \emptyset$.

On $\partial\Omega^{em}_h \times T$ the condition is imposed that the magnetic field $H$ field be normal to the boundary, and thus

$$n \times H = 0. \quad (4.15)$$

This kind of BC is called the natural boundary condition. In the vector potential notation, (4.15) reads

$$n \times \frac{1}{\mu} \nabla \times A = 0. \quad (4.16)$$

On $\partial\Omega^{em}_b \times T$ the condition is imposed that the magnetic flux density $B$ be tangent to the boundary, and thus

$$n \cdot B = 0. \quad (4.17)$$

This kind of BC is termed the essential boundary condition. In the case shown in Figure 4.2, essential BC have been imposed everywhere, that is, $\partial\Omega^{em} = \partial\Omega^{em}_b$. 
CHAPTER 4. COMPUTATIONAL APPROACH

Since the electromagnetic problem is defined on a bounded domain and since the above conditions are usually met in an only approximate way at the boundary of a bounded domain, the question arises as to reduce the boundary effects as much as possible without excessively increasing the computational domain. This is accomplished by means of a suitable space contraction that maps an unbounded domain to a bounded one \[79, 65, 58\]. Thanks to this kind of contraction, the relevant BC are sometimes termed far field boundary conditions. In the case shown in Figure 4.2, the relatively thin, outermost annulus (actually a hollow, truncated cylinder) in the air box is the portion of the domain $\Omega^{em}$ where a space contracting map is defined.

Owing to the low frequency approximation, two “static” problems have to be solved, namely the electrostatic and the magnetostatic one. No time derivative is involved in both problems. Nevertheless, the current boundary condition $i(t)$ for the electrostatic problem is time dependent, and so is also its relevant output field, i.e., $j(x,t)$. As a consequence also the forcing term of the magnetostatic problem is time dependent. This simply amounts to solving the electrostatic and the magnetostatic problems at many different time instants, with different input data, but with no internal differential constraint with previous or future time instants. In other words, the result is a sequence of independent static problems. Therefore, initial conditions are not relevant. With the conditions and assumptions provided, the electromagnetic problem is a sequence of several boundary value problems.

4.2.4 Formulation of the Electric Network Problem

The arcing circuit breaker which is the object of our research is part of a bigger electric network. A reliable simulation must account for the interaction between the arc, for which the energy is supplied by the network, and the network itself, which is on its turn influenced by the presence of the arc and, in particular, by the specific conductive state of the arc. A more structured and general approach to the problem is treated in §5.2. The same or similar reasoning also applies to the multiphysical approach to the arc. We shall now consider a special case, which is however sufficient to describe all of the network conditions that we have considered in our computational approach. The simplicity of the particular case will also allow a simpler understanding of the technique, without prejudicing the implementation in more complex and more general cases.

We assume that the arcing breaker is inserted as a serial component into an electric network also containing a voltage generator, a charged capacitor, a resistor, and an inductor. We consider the case of a series connection of the aforementioned components, so that a unique current $i(t)$ flows in the mesh of
the network. The governing equation (Kirchhoff’s mesh law) reads
\begin{equation}
    u_G(t) + u_C(t) = u_R(t) + u_L(t) + u(t),
\end{equation}
where $u_G(t)$ is the voltage supplied by the generator, $u_C(t)$ is the voltage supplied by the charged capacitor, $u_R(t)$ is the ohmic voltage drop due to the resistor, $u_L(t)$ is the voltage drop across the inductor and, finally, $u(t)$ is the arc voltage.

We consider a voltage supply from the generator of the form
\begin{equation}
    u_G(t) = V_{DC} + V_{AC}\sin(\omega t + \theta),
\end{equation}
where a constant and a sinusoidal source are added together. The constant $V_{DC} \in \mathbb{R}$ accounts for a constant voltage supply contribution. The rest of the r.h.s. accounts for an alternating contribution. The constant $V_{AC} \in \mathbb{R}^+$ is the amplitude of such a contribution, while $\theta \in \mathbb{R}$ is the phase shift and $\omega$ is the (angular) frequency. Each of the two contributions is switched off by selecting a null value of the relevant amplitude.

In order to detail the voltage from the discharging capacitor, we introduce the electric charge $q(t)$ flown away from the capacitor banks, from an initial time $t_0$ to the current time $t$. It reads
\begin{equation}
    q(t) := \int_{t_0}^{t} i(t') \, dt'.
\end{equation}
The symbol $q(t)$ must not be confused with the heat flux $q(x, t)$. The voltage supplied by the capacitor is
\begin{equation}
    u_C(t) = u_C(t_0) - \frac{q(t)}{C},
\end{equation}
where $u_C(t_0) \in \mathbb{R}$ is the initial voltage on the capacitor banks at time $t_0$ and where $C \in \mathbb{R}^+$ is the capacitance. The ohmic voltage drop across the resistor is
\begin{equation}
    u_R(t) = Ri(t),
\end{equation}
where $R \in \mathbb{R}^+$ is the resistance. The voltage drop due to the inductor is
\begin{equation}
    u_L(t) = L \frac{di(t)}{dt},
\end{equation}
where $L \in \mathbb{R}^+$ is the inductance.

Finally, the arc voltage drop $u(t)$ is the output from the magnetohydrodynamic problem, which is of distributed nature and is addressed by means of the
fluid dynamic, radiative transfer and electromagnetic problems. Particularly, $u(t)$ is the voltage difference in between the electric terminal $\partial \Omega_{em}$, where a current density distribution is imposed, and the electric terminal $\partial \Omega_{em}'$, where a reference (null, for the sake of convenience) voltage is imposed. The network current $i(t)$, coinciding with the arc current, and the arc voltage $u(t)$ realize the coupling in between the lumped parameter and the distributed parts of the electric network.

By plugging the voltage contributions (4.19), (4.21), (4.22), (4.23) and expression (4.20) into (4.18), an integro-differential equation is obtained, in the only unknown $i(t)$. As a matter of fact, $u(t)$ has to be considered known when solving the electric network, in the spirit of weak coupling.

4.3 Problem Solution

In section §4.2 we have introduced the “exact” problem to solve. Of course the term “exact” is actually up to a multitude of approximations of physical nature; see chapter §3. In this section we consider a possible numerical approximation of such a mathematical problem, which is too difficult to admit an analytical, closed-form solution.

4.3.1 The Coupling Procedure

The set of problems described in previous sections are coupled and should be solved as one problem (strong coupling). Fortunately enough, it is possible to apply a divide et impera approach. The main idea is to iteratively go through a loop in which the problems are solved separately and sequentially (weak coupling). Each problem provides input data for the next problem and receives input data from the previous one, where “next” and “previous” are from the logical and not temporal point of view.

In what follows we briefly outline the weak coupling procedure. The following diagram is intended to provide a visual help during the discussion.
4.3. PROBLEM SOLUTION

The interaction of the different problems actually involves many small loops. A first loop is the one between the electric network, as a lumped parameter problem, and the distributed electromagnetic problem. The network provides at each time step a current \(i(t)\) to the distributed problem (which is cast into a boundary condition on \(\partial \Omega^m_{em}\), specifying the prescribed current density), which returns back the voltage drop \(\Delta u(t) = u(t) - 0 = u(t)\) (since a null reference voltage has been set on \(\partial \Omega^m_u\) and where \(u(t)\) is the average voltage on \(\partial \Omega^m_{em}\) which is required to make that current flow in the arc channel of the breaker.

The second loop involves the distributed electromagnetic problem and the fluid dynamic problem. The former provides the current density field \(j(x,t)\), the electric field \(E(x,t)\) and the magnetic flux density field \(B(x,t)\), getting back in return the temperature field \(T(x,t)\) and the pressure field \(p(x,t)\), with which electrical conductivity \(\sigma(x,t)\) may be updated. The third loop involves the fluid dynamic problem and the radiative transfer problem. The former provides the temperature field \(T(x,t)\) and pressure field \(p(x,t)\), with which the radiative properties and the black body radiation is computed, getting back the radiative heat flux field \(q_r(x,t)\).

The electromagnetic problem, on its turn, is already decoupled in an electrostatic and in a magnetostatic problem, to be solved in sequence. The current \(i(t)\) is the boundary condition for the electrostatic problem, which outputs the electric field \(E(x,t)\) (obtained through the scalar potential field \(u(x,t)\)). Plugging the electric field and the electrical conductivity \(\sigma(x,t)\) into Ohm’s law, the current density field is obtained and passed to the magnetostatic problem, which outputs the magnetic flux density field \(B(x,t)\) (obtained through the vector potential field \(A(x,t)\)). The logical flow is illustrated by the following diagram.
4.3.2 Numerical Approximation of Differential Problems

The differential problems governed by PDE or ODE are solved in an approximate way by means of suitable discretization methods for the spatial differential operator (in case of PDE) and for the discretization of the temporal differential operator (both in the case of PDE and of ODE). All of such methods represent well-known and consolidated approaches.

With reference to a bounded computational domain $\Omega \subset \mathbb{R}^3$ and a time period $T \in \mathbb{R}$, we consider a rather generic initial boundary value problem, governed by the archetypal PDE

$$\mathcal{D}_t(u) + \mathcal{D}_x(u) = \mathcal{F},$$

(4.24)

to hold $\forall(x, t) \in \Omega \times T$. In (4.24), $u : \Omega \times T \to \mathbb{R}$ is an unknown function of spatial coordinates $x$ and of time $t$; $\mathcal{D}_t : C^1(T) \to C^0(T)$ is a temporal differential operator, only involving first partial derivatives with reference to time; $\mathcal{D}_x : C^h(\Omega) \to C^0(\Omega)$ is a spatial differential operator, only involving partial derivatives with reference to spatial coordinates and up to order $h$; and finally $\mathcal{F} : \Omega \times T \to \mathbb{R}$ is a known function, gathering the volumic forcing terms. The problem is supplemented by suitable boundary conditions, gathering surfacic forcing terms, and by suitable initial conditions. As regards the latter,
a Cauchy initial value problem is always the case in our problems. Boundary conditions are problem specific and therefore too variegated to allow a description to be compact and comprehensive at the same time. For this reason we shall treat this topic when dealing with the specific problems themselves. Even if, in general, PDE can be much more complex than assumed above, the structure of (4.24) suffices to cover the differential problems found in the proposed approach to MHD.

The spatial differential operator $\mathcal{D}_x$ needs not be linear, as in the case, e.g., of Navier-Stokes equations, while we can always assume that the temporal differential operator be linear (and, as a matter of fact, $\mathcal{D}_t$ is essentially $\partial_t$). The nonlinear nature of the spatial differential operator (when applicable) makes nonlinear the whole differential problem. Unfortunately this prevents from easily proving existence and uniqueness of solution, and its continuous dependence from input data. Particularly, Navier-Stokes equations, which are only a portion of the MHD problem, are still lacking of proofs for basic properties such as the existence of smooth solutions in the 3D case, and with bounded kinetic energy (the Navier-Stokes existence and smoothness problem [38]). This is probably one of the biggest concerns when approaching MHD, which, as a consequence, is a problem far from being fully dominated from the mathematical standpoint.

Since the governing PDE (4.24) must hold everywhere in space and time (i.e., a very strong requirement, from the mathematical standpoint), it is called the strong form of the differential problem. A possible solution to the strong form of a differential problem must be sufficiently regular, that is, it must be differentiable at least for the required number of times so that all differential operators make sense in the context of classical analysis (i.e., without considering distribution theory). Precisely and owing to the above assumptions, $u$ must be of class $C^h(\Omega)$ as a function of space and of class $C^1(\mathcal{T})$ as a function of time. Apart from the practical difficulty of finding closed form solutions, even from the theoretical point of view such a regular functional setting is too restrictive, for it is easy to show simple cases of physical or engineering interest which do not admit solution in $C^h$ spaces.

To overcome this problem, the distributional theory of differential equations has been developed [78], with a wider and more practical functional setting. Particularly, a suitable functional space

$$V \subseteq H^k(\Omega)$$

(4.25)

is defined, the regularity index $k$ in Sobolev spaces $H^k(\Omega)$ being determined by the regularity index $h$ in classical spaces $C^h(\Omega)$. The exact definition of $V$ depends on the type of applied boundary conditions. The strong form is cast...
into the problem of seeking a solution \( u \in V \) such that
\[
\int_{\Omega} D_t(u) \cdot v \, d\Omega + \int_{\Omega} D_x(u) \cdot v \, d\Omega = \int_{\Omega} \mathcal{F} \cdot v \, d\Omega \tag{4.26}
\]
holds \( \forall v \in V \). Equation (4.26) is called the weak form of the differential problem and is obtained by multiplying (4.24) by a test (or trial) function \( v \) and integrating over the computational domain, side by side. The weak form is called like that because it implies a milder mathematical requirement than the relevant strong form, since the PDE is only asked to hold in a weighted average sense, the test functions being the weighting functions. As a consequence, the strong form implies the weak form, but in general not vice versa. In this way, the solution, regarding its spatial dependence, is sought for in a wider class of functions, in the framework of Sobolev spaces \( H^k(\Omega) \). This way, it is possible to enlarge the set of problems admitting a solution. A “good” and “reasonable” choice of the test space \( V \) guarantees from being over-tolerant, that only “good” and “reasonable” solutions have become admissible.

The weak form (4.26) is still an infinite dimensional problem, for so is the search and test space \( V \). To find a numerical approximation to the solution, the space \( V \) is substituted by an approximating, finite dimensional subspace \( V_n \subseteq V \) such that \( \dim V_n = n < \dim V = \infty \). The solution which is now sought for is an approximation \( u_n \) of \( u \), lying in \( V_n \), and the weak form condition (4.26) is tested with reference to \( V_n \) functions only. The problem is thus to seek a solution in \( u_n \in V_n \) such that
\[
\int_{\Omega} D_t(u_n) \cdot v \, d\Omega + \int_{\Omega} D_x(u_n) \cdot v \, d\Omega = \int_{\Omega} \mathcal{F} \cdot v \, d\Omega \tag{4.27}
\]
holds \( \forall v \in V_n \). The quality of the approximation, and thus of the final numerical solution, strongly depends on the choice of the approximating subspace \( V_n \). If the choice of the approximating subspace is smart enough, then a refinement of the approximation \( u_n \) is obtained for increasingly higher values of \( n \). A desirable condition is obviously that \( u_n \to u \) for \( n \to \infty \), that is, to work with a convergent method.

Usually, a suitable finite sequence of functions \( \{\psi_i(x)\}_{i=1}^n \), is introduced, where the \( \psi_i : \Omega \to \mathbb{R} \) are functions of the spatial coordinates. Then, \( V_n \) is chosen as the subspace spanned by such functions, that is,
\[
V_n := \langle \psi_i \mid i \in \{1, \ldots, n\} \rangle. \tag{4.28}
\]
A good selection of the \( \psi_i \) is such not to introduce linear dependent functions, which would be of no use in spanning the subspace. Since the weak form condition (4.26) is linear in \( v \), one can also profitably test it with the only
4.3. PROBLEM SOLUTION

reference to the subspace generating functions. Since $u_n \in V_n$, then it can be expressed in the form

$$u_n(x, t) = \sum_{j=1}^{n} u_j(t) \psi_j(x),$$  \hspace{1cm} (4.29)

where the $u_j : \mathcal{T} \rightarrow \mathbb{R}$ are functions of time. In this way, the problem becomes finite dimensional in the physical space, time dependence being still continuous. The problem is now to seek $n$ time functions $u_j(t)$ such that

$$\sum_{j=1}^{n} \mathcal{D}_t(u_j) \int_{\Omega} \psi_i \cdot \psi_j \, d\Omega + \int_{\Omega} \mathcal{D}_x \left( \sum_{j=1}^{n} u_j \psi_j \right) \cdot \psi_i \, d\Omega = \int_{\Omega} \mathcal{F} \cdot \psi_i \, d\Omega$$  \hspace{1cm} (4.30)

holds $\forall i \in \{1, \ldots, n\}$. The test functions $\psi_i$, which span the test space (i.e., the class of functions used to verify the weak form equation), are also named basis (or shape) functions, for they are used to span the search space (i.e., the class of functions where the solution to the weak form is sought for) and to express the approximated solution $u_n$. The above variational approach for space discretization is called a Galerkin method when, like in this case, the search and test space coincide, together with shape and test functions.

Usually, the shape functions are defined with reference to a suitable partition of the computational domain into a finite collection of elements (or cells) $\Omega_i$ such that

$$\Omega = \bigcup_i \Omega_i$$  \hspace{1cm} (4.31)

and such that the intersection $\Omega_i \cap \Omega_j$ between any two distinct elements is either empty or it is a portion of their boundary. In other words, $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$, where $\bar{\Omega}_i := \Omega_i \setminus \partial \Omega_i$ is the interior of the element. The partition is termed the computational mesh (or grid) and it is indicated as $\mathcal{G}$. The vertices of the grid are termed nodes and they are collected into the set $\mathcal{N}(\mathcal{G})$. In the case of the finite element method, other nodes than the vertices could be included in $\mathcal{N}(\mathcal{G})$, namely mid-side nodes or other points on the edges, or on the facets, or also in the cell interior. With $\mathcal{N}_0(\mathcal{G})$ we shall refer to the subset of $\mathcal{N}(\mathcal{G})$ without nodes falling on the portion (if any) of $\partial \Omega$ where Dirichlet BC are imposed. Similarly, the elements of the grid are collected into the set $\mathcal{C}(\mathcal{G})$ and the edges are collected into the set $\mathcal{E}(\mathcal{G})$. Shape functions are then defined in a implementation specific way, as later detailed with reference to each particular problem. Notice that the requirements on the partition do not impose, in the general case, that the mesh be conformal, that is, that $\Omega_i \cap \Omega_j$ be either empty or a common face, or a common edge, or a common vertex. In non conformal grids, two neighboring cells may share only a common portion of two (or more) faces or edges. In conformal grids each node is necessarily a vertex of all of the elements it belongs to as a point.
What is left after space discretization is a (large) system of ODE. Time discretization is the final step toward an algebraic problem. This issue will be discussed with reference to each single problem. A time marching scheme is always addressed, starting from the initial conditions (Cauchy initial value problem). In what follows, as regards time discretization, we shall denote with (an additional, if others are already present) subscript \( k \) the value of a quantity when evaluated at time \( t_k \) (i.e., \( \circ_k = \circ(t_k) \), where \( \circ \) stands for a generic, time dependent quantity). We consider the time step from \( t_k \) to \( t_{k+1} = t_k + \Delta t_k \).

The time step size \( \Delta t_k \) has subscript \( k \) so to allow for adaptive time stepping (either as a consequence of an automatic procedure or as a user choice). When solving such a time step, everything at time \( t_k \) or antecedent has to be assumed known, from the solution to the previous time steps, while everything at time \( t_{k+1} \) is still unknown.

### 4.3.3 Discretization of the Fluid Dynamic Problem

The fluid dynamic problem (see §4.2.1) is discretized in space by means of the finite volume method (FVM) [73, 161]. This choice is due to the large availability of finite volume codes for computational fluid dynamics. We have utilized commercial code Fluent as CFD solver. A better approach could be the finite element method (FEM) [121] or the spectral element method (SEM) [20], which can profitably replace the FVM without changing the spirit of our reasonings. The coupled set of mass, momentum and energy equations are solved by means of the classical density based algorithm for CFD.

We hereafter provide a short account of space discretization by means of the FVM, in the framework of Galerkin method; see §4.3.2. We consider the general transport equation, which can be taken as the archetype PDE for the mass, momentum and energy conservation laws. All of the reasonings discussed in such a general frame also hold in the specific context of the three conservation principles. Rewriting the general transport equation (3.34) with reference to the plasma velocity \( \mathbf{u} \) instead of the \( s \)-th species velocity \( \mathbf{u}_s \), one gets the strong form (4.24), which reads

\[
\frac{\partial \rho g}{\partial t} + \nabla \cdot (\rho g \mathbf{u}) - \nabla \cdot \Gamma \nabla g = S_g, \tag{4.32}
\]

to hold \( \forall \mathbf{x} \in \Omega^f \) and \( \forall t \in \mathcal{T} \). In the case of (4.32), \( \rho, \mathbf{u} \) and \( \Gamma \) should be of class \( C^1(\Omega^f) \), \( g \) should be of class \( C^2(\Omega^f) \). As functions of time, all should be at least of class \( C^0(\mathcal{T}) \), while \( \rho \) and \( g \) must be of class \( C^1(\mathcal{T}) \).

To get the weak form (4.26), a suitable functional space (4.25) is defined, namely \( V^f \subseteq H^1(\Omega^f) \), and the strong form is cast into the problem of seeking
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a solution in $V_{fd}$ such that

$$
\int_{\Omega_{fd}} \left( \frac{\partial \rho g}{\partial t} + \nabla \cdot (\rho g u) - \nabla \cdot (\Gamma \nabla g) \right) \cdot v \, d\Omega = \int_{\Omega_{fd}} S_g \cdot v \, d\Omega \quad (4.33)
$$

holds $\forall v \in V_{fd}$. In order to define a finite dimensional, approximating functional space (4.28), a finite volume mesh (4.31) is introduced. A sample finite volume computational grid is shown in Figure 4.4, with reference to the same model shown in Figure 4.1. The mesh is constituted of approximately 180 000 cells and 40 000 nodes. In the context of the FVM, $n$ will refer to the number of cells in the mesh, i.e., $n := |C(G)|$. Only tetrahedral cells have been used in our problems, even though also differently shaped elements could be used. All the grids we have considered are conformal. An adaptive, local, conformal mesh refinement and coarsening may be used, based on the intensity of the temperature field, so to increase the accuracy of the solution by treating

Figure 4.4: Sample computational grid for the fluid dynamic and for the radiative transfer problem.
with more detail the region where the arc stands. Non conformal grids and refinements/coarsenings could also be conceived, in theory.

For each cell in $C(G)$, we consider the characteristic function of the set of its points, that reads

$$\chi_i(x) = \begin{cases} 
1 & \text{if } x \in \Omega_i \\
0 & \text{otherwise.}
\end{cases} \quad (4.34)$$

The characteristic functions are chosen as test functions, that is, $\psi_i := \chi_i$, $i \in \{1, \ldots, n\}$, and the finite dimensional approximating subspace (4.28) reads

$$V_n^{fd} := \langle \chi_i \mid i \in \{1, \ldots, n\} \rangle. \quad (4.35)$$

Therefore, the formulation (4.30) in the case of Galerkin-FVM method leads to the problem of seeking a solution to

$$\int_{\Omega_i} \left( \frac{\partial \rho g}{\partial t} + \nabla \cdot (\rho g \mathbf{u}) - \nabla \cdot (\Gamma \nabla g) \right) \cdot \chi_i \, d\Omega = \int_{\Omega_i} S_g \cdot \chi_i \, d\Omega, \quad (4.36)$$

holding $\forall i \in \{1, \ldots, n\}$. The choice of element characteristic function reveals useful in order to simplify the equations to solve. As a matter of fact, they act like an element-pass filter, so that (4.36) reduces to

$$\int_{\Omega_i} \left( \frac{\partial \rho g}{\partial t} + \nabla \cdot (\rho g \mathbf{u}) - \nabla \cdot (\Gamma \nabla g) \right) \, d\Omega = \int_{\Omega_i} S_g \, d\Omega, \quad (4.37)$$

to hold $\forall i \in \{1, \ldots, n\}$, that is, on any cell in $C(G)$. Applying the divergence theorem A.5, one gets the familiar form

$$\int_{\Omega_i} \frac{\partial \rho g}{\partial t} \, d\Omega + \int_{\partial \Omega_i} \rho g \mathbf{u} \cdot \mathbf{n} \, d\Gamma - \int_{\partial \Omega_i} (\Gamma \nabla g) \cdot \mathbf{n} \, d\Gamma = \int_{\Omega_i} S_g \, d\Omega, \quad (4.38)$$

to hold $\forall i \in \{1, \ldots, n\}$. This equation looks extremely simple because it expresses the conservation law (4.32) in integral terms over a “control volume” $\Omega_i$. From left to right, one finds the storage term as a volume integral, the convective term as a surface integral, the diffusive term as a surface integral and the source term as a volume integral.

The very natural equation (4.38) is the traditional entry point to introduce the FVM and explains the historical deduction of such a method as the direct and discretized application of the conservation laws over a mesh of control volumes. This consideration ends the list of positive remarks on the FVM. From this point on, mathematical elegance is unfortunately lost and a series of compromise choices need being taken in order to get to the final, discrete formulation. Owing to the expression (4.29) of the approximate solution, in
the Galerkin-FVM method the latter is straightforwardly and naturally defined inside each cell, that is,

\[ g_n(x, t) = \sum_{i=1}^{n} g_i(t) \chi_i(x), \quad \forall x \in \bigcup_{i=1}^{n} \Omega_i, \]  

(4.39)

and similarly for \( \rho \) and \( u \). In other terms, the approximating solution is assumed element wise constant, with discontinuity jumps occurring in general from each element to the neighboring ones. The cell value \( g_i \) (and similarly for other quantities) is usually referred to as belonging to the cell centroid \( x_i \), even though this is merely a convention required to locate precisely a value in space. Also all other spatial functions possibly enclosed inside \( S_g \) are approximated straightforwardly, for volume integrals need only to be computed. The problem regards surface integrals, since element characteristic functions \( \chi_i \) are there discontinuous and which value of the approximated quantity is to be used is a questionable choice. Whatever the choice, the unknowns remain cell centroid values and facet values must be expressed in terms of cell centroid values. Postponing the problem for a while, we assume that each element \( \Omega_i \) has a polyhedral shape, with a number \( f_i \) of facets, possibly different element wise, and we term \( g_{ij} \) the value of \( g \) on the \( j \)-th facet of the \( i \)-th cell. An analogous notation is used for \( \rho \) and \( u \) and all other quantities. In case of conformal grids, also element values at the surface are assumed to be facet wise constant and referred to as belonging to the facet centroid. In the non conformal case one can proceed with portions of facets. There is no guarantee that such values be equal over the common facet of two neighboring cells. Depending on the way values at the surface are defined, this internal incoherence may arise.

Owing to the above assumptions and introducing the element volume \( |\Omega_i| \) and the facet area \( |\Gamma_{ij}| \) (not to be confused with diffusivity; mind the boldface symbol), the weak form (4.38) is finally discretized as

\[ \frac{\partial \rho_i g_i}{\partial t} |\Omega_i| + \sum_{j=1}^{f_i} \rho_{ij} g_{ij} u_{ij} \cdot n_{ij} |\Gamma_{ij}| - \sum_{j=1}^{f_i} (\Gamma_{ij} (\nabla g)_{ij}) \cdot n_{ij} |\Gamma_{ij}| = S_{gi} |\Omega_i|, \]  

(4.40)

to hold \( \forall i \in \{1, \ldots, n\} \).

The problem now arises as to suitably define facet values in terms of cell centroid values. As anticipated, this is a rather tricky and technical issue, far from mathematical elegance. Actually, many alternative possibilities exist, other than the one we have used. We chose the simplest one, but CFD codes (including Fluent) also allow finer choices. It is difficult to predict the opportunity of a choice or another, especially when adopted in such a complex problem like MHD. For the sake of simplicity we consider the case of cells bordering by a complete common facet, like in a conformal mesh. For the sake of brevity, we
shall use subscript \( j \) to refer to the cell centroid value of the only neighboring cell with the \( j \)-th facet of the \( i \)-th cell. In that case, we shall also refer to the neighboring cell as the \( j \)-th one. The non conformal case is formally more intricate but conceptually without additional complexity. Simply, one must sub-partition the facets so to work with common portions. Needless to say, this is definitely inelegant from the mathematical standpoint.

As regards the convective term, in the otherwise unstable [121] convection-diffusion problem, upwinding is known to be an opportune method to bring values from cell centroids to facet centroids. We use first-order upwind, that is, to assign a cell centroid value to the facet centroid, directly and without further elaboration. For a generic quantity \( \varphi \), which stands for \( \rho, g \) and \( u \) (component wise), the first-order upwind scheme reads

\[
\varphi_{ij} = \begin{cases} 
\varphi_i & \text{if } u_i \cdot \mathbf{n}_{ij} > 0 \\
\varphi_j & \text{if } u_i \cdot \mathbf{n}_{ij} < 0.
\end{cases}
\]  

(4.41)

The cell value of the upstream cell is always used: if the flow is from the \( i \)-th cell to the \( j \)-th cell, then the upstream cell is the \( i \)-th, and vice versa. Needless to say, if \( u_i \cdot \mathbf{n}_{ij} = 0 \), i.e., if the face is tangent to the flow, then that face is not contributing to the net outflow and, as a consequence, it does not contribute to the summation of the convective term.

As regards the diffusive terms, gradients are needed on the facets. The simple choice to use the difference quotient

\[
(\nabla g)_{ij} = \frac{g_j - g_i}{|x_j - x_i|}
\]  

(4.42)

is second-order accurate on a uniform grid, like vast regions of the one shown in Figure §4.4, and first-order accurate in the worst case of a non uniform grid with high size change. Also, the diffusivity tensor is needed on the facets and a simple choice is the harmonic mean

\[
\varphi_{ij} = \frac{2\varphi_i \varphi_j}{\varphi_i + \varphi_j},
\]  

(4.43)

where \( \varphi \) stands for \( \Gamma \), component wise.

The time discretization of the fluid dynamic problem may be outlined as follows with reference to (4.40) and the \( i \)-th cell in \( \mathcal{C}(G) \). Including all the spatial dependence - actually the relevant discretized counterpart - in a functional \( F_i \), defined as

\[
g \mapsto F_i(g) := S_{gi} + \sum_{j=1}^{f_i} \left( \Gamma_{ij} (\nabla g)_{ij} - \rho_{ij} g_{ij} u_{ij} \right) \cdot \mathbf{n}_{ij} \right) \cdot |\Gamma_{ij}| / |\Omega_{ij}|,
\]
the time continuous problem to solve reads
\[
\frac{\partial}{\partial t}(\rho_i g_i) = F_i(g).
\] (4.44)

Then a time discretized counterpart of (4.44) is obtained by adopting the implicit scheme
\[
\frac{3(\rho_i g_i)_{k+1} - 4(\rho_i g_i)_k + (\rho_i g_i)_{k-1}}{2\Delta t_k} = F_i(g_{k+1}),
\] (4.45)

which is second-order accurate and unconditionally stable. More precisely, the scheme (4.45) is the classical second order backward differentiation formula (BDF) [57]. The typical time step for the fluid dynamic problem is of the order of 1\,\mu s. Some thousands time steps are required to complete a whole simulation.

This completes the spatial and temporal discretization of the general transport equation, from an initial-boundary value differential problem to a very large set of algebraic equations. The discretized counterpart of the fluid dynamic problem is closed by applying the above method to all of the occurrences of the general transport equation in system (4.1), by accounting for the other, non differential, internal constraints between the problem variables and treating problem-external quantities in the spirit of weak coupling.

Since the fluid dynamic problem is nonlinear, both for the solution dependent physical properties and for the quadratic \(u \otimes u\) term, the solution to each time step requires solving a system of nonlinear equations of the type
\[
3(\rho_i g_i)_{k+1} = 2\Delta t_k \cdot F_i(g_{k+1}) + 4(\rho_i g_i)_k - (\rho_i g_i)_{k-1}.
\] (4.46)

Nonlinearity is tackled with a sequential procedure of linearization centered on a tentative solution, solution to the linearized problem and update of the tentative solution. The physical properties are updated iteration wise. The above cycle constitutes an outer looping (as an inner looping will be introduced shortly hereafter), which is executed until convergence. One can synthetically describe each outer iteration as the solution to the linear system
\[
L_{k+1,j}^{fd} u_{k+1,j}^{fd} = f_{k+1,j}^{fd},
\] (4.47)

where \(u_{k+1,j}^{fd}\) is the vector collecting the unknown fluid dynamic variables to be solved for (that is, mass density, fluid velocity and temperature, all in cell centroids), \(L_{k+1,j}^{fd}\) is the coefficient matrix and \(f_{k+1,j}^{fd}\) is the known vector term. Subscript \(k + 1\) refers to the number of the time step. Subscript \(j\) refers to the number of the outer, nonlinear iteration. The matrix and the vector of the system are different time step wise and iteration wise, in general.
The system (4.47) is solved by means of the under-relaxed Gauss-Seidel method, which introduces an inner looping. The additive splitting

\[ \mathbf{L}^{fd}_{k+1,j} = \mathbf{M}^{fd}_{k+1,j} - \mathbf{N}^{fd}_{k+1,j} \] (4.48)

is introduced for the coefficient matrix, where \( \mathbf{M}^{fd}_{k+1,j} \) is a diagonal matrix equal to the diagonal of \( \mathbf{L}^{fd}_{k+1,j} \), whilst \( -\mathbf{N}^{fd}_{k+1,j} \) collects the off-diagonal terms. Owing to the splitting (4.48), the system (4.47) is immediately and equivalently rewritten as

\[ \mathbf{M}^{fd}_{k+1,j} \mathbf{u}^{fd}_{k+1,j} = \mathbf{N}^{fd}_{k+1,j} \mathbf{u}^{fd}_{k+1,j} + \mathbf{f}^{fd}_{k+1,j}. \] (4.49)

Gauss-Seidel looping consists in approximating the sought for solution \( \mathbf{u}^{fd}_{k+1,j} \) by means of a sequence of tentative solutions \( \mathbf{u}^{fd}_{k+1,j,h} \), where the subscript \( h \) refers to the number of the inner iteration. The tentative solution is updated according to the scheme

\[
\mathbf{u}^{fd}_{k+1,j,h+1} = \omega \cdot \left[ (\mathbf{M}^{fd}_{k+1,j})^{-1} \mathbf{N}^{fd}_{k+1,j} \mathbf{u}^{fd}_{k+1,j,h} + (\mathbf{M}^{fd}_{k+1,j})^{-1} \mathbf{f}^{fd}_{k+1,j} \right] \\
+ (1 - \omega) \cdot \mathbf{u}^{fd}_{k+1,j,h},
\] (4.50)

where the first addendum is the solution to (4.49) in the hypothesis that the r.h.s. has not been updated yet, for the previous inner iterate is retained, whilst the second addendum is the tentative solution at the previous inner iterate. The inversion of matrix \( \mathbf{M}^{fd}_{k+1,j} \) is harmless, for it is diagonal. The under-relaxation factor (URF) \( \omega \in (0, 1] \) realizes a convex combination of the two addenda. If \( \omega = 0 \), the tentative solution would not be updated and the iteration would stall. Small values of \( \omega \) are thus a conservative choice which may help convergence, even though a slow rate of convergence has to be expected. On the other hand, values of \( \omega \) close to 1 represent an aggressive choice, which privileges the novelty of the update and neglects the previous iterate. If convergence is attained, a higher rate of convergence is achieved.

Inner, Gauss-Seidel looping is initialized based on the last converged outer, nonlinearity iteration, that is, \( \mathbf{u}^{fd}_{k+1,j+1,0} = \mathbf{u}^{fd}_{k+1,j,h_j} \), where \( h_j \) is the number of inner, Gauss-Seidel iterations used at the \( j \)-th outer, nonlinearity iteration. Outer, nonlinearity looping is initialized based on the previous time step solution, that is, \( \mathbf{u}^{fd}_{k+1,0,0} = \mathbf{u}^{fd}_{k,jk,h_{jk}} \), where \( j_k \) is the number of outer, nonlinearity iterations used at the \( k \)-th time step. Finally, the whole time stepping scheme is initialized based on initial conditions, that is, \( \mathbf{u}^{fd}_{0,0,0} = \mathbf{u}^{fd}|_{t=0} \), whence the importance of a good initial guess of the flow pattern (which is usually not simple) in order to achieve both convergence and a good convergence rate (at least initially).
Gauss-Seidel is an iterative method for linear systems. Therefore, it must be preconditioned in order to solve large scale problems, such as those occurring in MHD simulations of electric arcs, in a reasonable number of (inner) iterations. An algebraic multigrid (AMG) method [64] is used as a preconditioner. The basic idea is to project, by a suitable restriction operator, the error in the solution to the linear system onto a coarser grid and to seek for a correction accounting for such an error. The solution to the coarser problem is then prolonged back to the finer problem, where it is used to improve the global solution. Restriction onto a coarser grid allows quickly removing lower frequency components of the numerical error. A multitude of sub-levels may be defined, hierarchically, and the restriction and prolongation procedure may be applied recursively. The hierarchy of sub-level grids is produced by clustering neighboring cells. The method is suited for free, or unstructured, grids, because clustering is accomplished by means of algebraic considerations of the system matrix and without other geometrical data except for the information about cell neighbors, whence the denomination of algebraic, in contrast with geometric multigrid methods for structured grids.

4.3.4 Discretization of the Radiative Transfer Problem

The radiative transfer problem is also discretized in space by means of the finite volume method and commercial code Fluent is still used to solve the equations. This choice is bound the choice of the fluid dynamic solver, since the latter usually enables the approximated solution to the RTE through the $P_1$ model and over the same computational grid used to solve the CFD problem; see Figure 4.4. This is because Helmholtz equation (4.2) is (loosely speaking) a special case of the general transport equation (3.34), with the storage and the convective terms dropped, with black body radiation as the source term and with the addition of a non differential, solution-proportional term. Also in the case of the RTE, the Galerkin-FEM or the Galerkin-SEM could be profitably employed, even though a real benefit would only be gained if the same solver and grid are used for both the fluid dynamic and the radiative transfer problem.

Particularly, the weak form (4.26) of the $b$-th band RTE (4.2) consists in seeking $G_b \in V^{rt}$ such that

$$
\int_{\Omega^{rt}} (\nabla \cdot \Gamma_b \nabla G_b) \cdot v \, d\Omega = \int_{\Omega^{rt}} \alpha_b (G_b - G_b^{bb}) \cdot v \, d\Omega 
$$

holds $\forall v \in V^{rt}$, where $V^{rt} := H^1(\Omega^{rt})$ is the test and search space (4.25). Proceeding as in the case of the general transport equation, the approximating space (4.28) is defined with reference to element characteristic functions (4.34),
CHAPTER 4. COMPUTATIONAL APPROACH

and

\[ V^n_r := \langle \chi_i \mid i \in \{1, \ldots, n\} \rangle. \quad (4.52) \]

Since the same computational grid is used both for the fluid dynamic problem and the radiative transfer one, then \( V^n_r = V^n_{fd} \). The Galerkin-FVM method leads to the problem of seeking a solution to

\[
\int_{\Omega^n_r} (\nabla \cdot \Gamma_b \nabla G_b) \cdot \chi_i \, d\Omega = \int_{\Omega^n_r} \alpha_b (G_b - G^{bb}_b) \cdot \chi_i \, d\Omega, \quad (4.53)
\]

holding \( \forall i \in \{1, \ldots, n\} \) and \( \forall b \in B \). Following the same steps as in the case of the fluid dynamic problem, each characteristic function reduce the integrals over the relevant cell and the divergence theorem A.5 is used to handle the diffusive term and to obtain a surfacic counterpart. One easily gets

\[
- \int_{\partial \Omega_i} (\Gamma_b \nabla G_b) \cdot \mathbf{n} \, d\Gamma + \int_{\Omega_i} \alpha_b G_b \, d\Omega = \int_{\Omega_i} \alpha_b G^{bb}_b \, d\Omega. \quad (4.54)
\]

With the same notation introduced in §4.3.3, and particularly expressing the approximate solution as

\[
G_{bn}(x,t) = \sum_{i=1}^{n} G_{bi}(t) \chi_i(x), \quad \forall x \in \bigcup_{i=1}^{n} \hat{\Omega}_i, \quad (4.55)
\]

then the weak form is finally discretized as

\[
- \sum_{j=1}^{f_b} (\Gamma_{bij} (\nabla G_b)_{ij}) \cdot \mathbf{n}_{ij} \mid_{\Gamma_{ij}} + \alpha_{bi} G_{bi} \mid_{\Omega_i} = \alpha_{bi} G^{bb}_{bi} \mid_{\Omega_i}, \quad (4.56)
\]

to hold \( \forall i \in \{1, \ldots, n\} \) and \( \forall b \in B \). In (4.56), the radiative diffusion coefficient \( \Gamma_{bij} \) on the \( j \)-th facet of the \( i \)-th cell must not be confused with the area \( \Gamma_{ij} \) of such a facet. The same methodologies are adopted as in the fluid dynamic problem to estimate facet centroid values from cell centroid values.

With reference to the \( b \)-th spectral band and collecting the problem unknowns \( G_{bi} \) into a vector \( u_{rt}^b \), the set of equations (4.56) may be rewritten as a collection of linear systems (one for each band) of the form

\[ \mathbf{L}_{rt}^b \mathbf{u}_{rt}^b = \mathbf{f}_{rt}^b, \quad \forall b \in B, \quad (4.57) \]

where the coefficient matrix \( \mathbf{L}_{rt}^b \) depends on the physical radiative properties of the air-plasma, while the known vector term \( \mathbf{f}_{rt}^b \) contains black body radiation and is also depending on the radiative properties of the air-plasma.

The same time stepping for the fluid dynamic problem is used. No special requirement is due to time discretization, since no time derivative appears in
4.3. PROBLEM SOLUTION

(4.56). Simply, the problems (4.57) stemming from the equations (4.56) are solved at any time step with updated values of the physical properties and of black body radiation. Moreover, since the radiative transfer problem is solved by the same CFD solver Fluent, a very natural interrelationship is established with the fluid dynamic problem, and the same (outer) iteration scheme is used, with a iteration wise updating of the above quantities. Therefore, at the \( j \)-th iteration of the \((k+1)\)-th time step a collection of linear systems

\[
L_{b,k+1,j}^{rt} u_{b,k+1,j}^{rt} = f_{b,k+1,j}^{rt}, \quad \forall b \in B
\]

(4.58)

has to be solved, the subscripts having the same meaning as in the case of the fluid dynamic problem.

4.3.5 Discretization of the Electromagnetic Problem

The electrostatic problem is discretized in space by means of the standard, node-based, Galerkin-FEM. A conformal, entirely tetrahedral mesh is used, constituted of approximately 680 000 elements and 120 000 nodes in the case of the idealized but realistic MCB from ABB shown in Figure 4.2, where the mesh of the air box is not represented for the sake of a better comprehension. It could be also possible to use hexahedral or pentahedral elements, but the choice of tetrahedra allows for a fast and automatic mesh generation in the case of complex geometries, which is an important issue when industrial cases are concerned and free meshes are practically mandatory.

The weak form (4.26) of the electrostatic problem (4.7) consists in seeking \( u \in V^{es} \) such that

\[
\int_{\Omega^{em}} (\nabla \cdot \varepsilon \nabla u) \cdot v \, d\Omega = 0
\]

holds \( \forall v \in V^{es} \), where

\[
V^{es} := H^1_0(\Omega^{em}) = \{ v \in H^1(\Omega^{em}) \mid v|_{\partial\Omega^{em}_u} = 0 \}
\]

(4.60)

is the test and search space (4.25) and superscript \( es \) stands for electrostatics. In (4.60), \( v|_{\partial\Omega^{em}_u} \) is the trace of \( v \) on the Dirichlet portion \( \partial\Omega^{em}_u \) of the boundary, where a fixed, null potential is imposed. By means of Green’s formula for the Laplacian (see Theorem A.3), the spatial partial derivatives contained in the divergence operator may be transferred from the sought solution \( u \) to the test function \( v \), yielding

\[
\int_{\partial\Omega^{em}} \varepsilon \frac{\partial u}{\partial n} v \, d\Gamma - \int_{\Omega^{em}} \varepsilon \nabla u \cdot \nabla v \, d\Omega = 0.
\]

(4.61)

The surface integral may be simplified by recalling that \( \partial\Omega^{em} = \partial\Omega^{em}_u \cup \partial\Omega^{em}_i \cup \partial\Omega^{em}_j \). On \( \partial\Omega^{em}_u \) the test function \( v \) is null by definition (4.60), on \( \partial\Omega^{em}_i \) the
normal derivative $\partial u/\partial n$ of the solution is null for (4.11) and finally, on $\partial \Omega_{em}^j$, the Neumann BC (4.9) is directly imposed, yielding

$$\int_{\Omega_{em}} \varepsilon \nabla u \cdot \nabla v \, d\Omega = - \int_{\partial \Omega_{em}^j} \frac{\varepsilon}{\sigma} j v \, d\Gamma.$$  \hspace{1cm} (4.62)

For the sake of compactness, the symmetric, positive definite, bilinear form $L^{es} : H^1(\Omega_{em}) \times H^1(\Omega_{em}) \rightarrow \mathbb{R}$ may be introduced, defined as

$$L^{es}(u, v) := \int_{\Omega_{em}} \varepsilon \nabla u \cdot \nabla v \, d\Omega,$$  \hspace{1cm} (4.63)

Figure 4.5: Sample computational grid for the distributed electromagnetic problem. For the sake of a better comprehension, the air box mesh is not shown.
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together with the linear functional $f^{es} : H^1(\Omega^m) \to \mathbb{R}$, defined as

$$f^{es}(v) := - \int_{\partial\Omega^m} \frac{\varepsilon}{\sigma} j v \, d\Gamma.$$  \hfill (4.64)

This allows rewriting the weak form as the problem to seek $u \in V^{es}$ such that

$$L^{es}(u, v) = f^{es}(v), \quad \forall v \in V^{es}. \hfill (4.65)$$

The approximating subspace (4.28) is defined the canonical way for node-based, tetrahedral, finite elements. Particularly, one defines the set

$$P_r := \{ p \in \mathbb{R}[x, y, z] \mid \deg p \leq r \} \hfill (4.66)$$

of the polynomials in space coordinates with real coefficients and with (total) degree not exceeding $r$. Then, one chooses the approximating subspace to be the set of element wise polynomials, of degree not exceeding $r$ and automatically satisfying the homogeneous Dirichlet BC, where applicable, that is, 

$$V^{es}_n := \{ v \in C^0(\Omega^m) \mid v|_{\Omega_i} \in P_r, \forall \Omega_i \in \mathcal{C}(G), v|_{\partial\Omega^m} = 0 \}, \hfill (4.67)$$

where $n = |\mathcal{N}_0(G)|$, that is, the number of nodes in the computational grid (excluding those on $\partial\Omega^m$), is the index of the approximation. The nodal dependence of the functional space (4.67) is easily explained as follows. We consider the generic $i$-th node $N_i \in \mathcal{N}_0(G)$ and the collection of the elements $\{\Omega_h\}_{h \in \mathcal{H}_i} \subseteq \mathcal{C}(G)$, where $\mathcal{H}_i$ is a suitable index set, such that $N_i \in \Omega_h$. Over each element $\Omega_h$, $h \in \mathcal{H}_i$, one may uniquely define the Lagrange polynomial $\ell_{i}^{h,r}$ of degree $r$ and such that $\ell_{i}^{h,r}(N_j) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol. In order to do so, one fixes the degree $r$. If $r = 1$, as in our implementation, then $\mathcal{N}(G)$ must consist only of cell vertices and $\ell_{i}^{h,1}$ is linear in the case of tetrahedra (as in the case at hand). If $r = 2$, then also mid-side nodes have to be part of $\mathcal{N}(G)$, and so forth for higher order interpolations. Finally, the $i$-th shape function $\varphi_i$, associated to the $i$-th node $N_i$, is defined to be equal to $\ell_{i}^{h,r}$ on $\Omega_h$, $h \in \mathcal{H}_i$, and it is prolonged elsewhere as an identically null function. By construction, $\varphi_i \in C^0(\Omega^m)$. In other terms, for $r = 1$ and for tetrahedral elements, the graph of the $i$-th shape function is pyramid shaped, with vertex in $N_i$. For the shape of their graph, such functions are frequently called hat functions. More generally, finite elements adopting shape functions defined as above are termed Lagrangian elements, for Lagrange polynomials are ultimately used. Since polynomials of degree not exceeding $r$ are an additive group, it is immediate to verify that the approximating space is actually spanned by the above shape functions, that is,

$$V^{es}_n = \{ \varphi_i \mid i \in \{1, \ldots, n\} \}. \hfill (4.68)$$
The approximate solution $u_n \in V_{es}^n$, which is continuous by construction, is thus expressed in the form (4.29) as

$$u_n(x, t) = \sum_{j=1}^{n} u_j(t) \varphi_j(x). \quad (4.69)$$

We can define the matrix and vectors

$$L^{es} := [L^{es}(\varphi_i, \varphi_j)], \quad u^{es} := [u_i], \quad f^{es} := [f^{es}(\varphi_i)], \quad (4.70)$$

and test the weak form (4.65) with test functions $\varphi_i$. With straightforward passages, basically exploiting the bi-linearity of the form $L^{es}$, the linear system

$$L^{es} u^{es} = f^{es} \quad (4.71)$$

is obtained, consisting of $n$ equations in the $n$ unknowns $u_i$.

Also in the case of electrostatics, time is not appearing with a partial derivative and the system (4.71) is simply solved at any time step with updated values of the electrical conductivity $\sigma$ and of the current density $j$ imposed on $\partial \Omega^{em}_j$. Formally, the system (4.71) is written as

$$L^{es} u^{es}_k = f^{es}_k, \quad (4.72)$$

where the subscript $k$ refers to the number of the time step to solve. No reference appears to additional iterations, for the problem is linear and there is no need for following the iterative scheme of CFD. The coefficient matrix brings no reference to the time step since the air-plasma permittivity is assumed constant.

The magnetostatic problem is discretized in space by means of the Galerkin-FEM, in the “edge element” version. The same conformal, entirely tetrahedral mesh used for electrostatics is retained also in the case of magnetostatics. Also in the case of edge elements, it could be possible to use hexahedral or pentahedral elements, but tetrahedra are preferred for the very same practical issue of automatic grid generation in the case of industrial, complex geometries such as the case shown in Figure 4.2, where the possibility to create a structured mesh is too time consuming and thus practically excluded.

The weak form (4.26) of the magnetostatic problem (4.14) consists in seeking $A \in V^{ms}$ such that

$$\int_{\Omega^{em}} \left( \nabla \times \frac{1}{\mu} \nabla \times A \right) \cdot v \, d\Omega = \int_{\Omega^{em}} j \cdot v \, d\Omega \quad (4.73)$$
holds $\forall \mathbf{v} \in \mathbf{V}^{ms}$, where

$$
\mathbf{V}^{ms} := \mathbf{H}_0(\text{curl}; \Omega^em)
$$

(4.74)

is the test and search space (4.25) and superscript $ms$ stands for magnetostatics. In (4.74), $\mathbf{n} \times \mathbf{v}|_{\partial \Omega^em}$ is the trace of $\mathbf{n} \times \mathbf{v}$ on the $\partial \Omega^em$ portion of the boundary, where a tangential magnetic flux density is imposed. By means of Green’s formula for the curl-curl operator (see Theorem A.4), the spatial partial derivatives contained in the outer curl operator may be transferred from the sought solution $\mathbf{A}$ to the test function $\mathbf{v}$, yielding

$$
\int_{\Omega^em} \frac{1}{\mu} (\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) \, d\Omega + \int_{\partial \Omega^em} \left( \mathbf{n} \times \frac{1}{\mu} \nabla \times \mathbf{A} \right) \cdot \mathbf{v} \, d\Gamma
$$

(4.75)

$$
= \int_{\Omega^em} \mathbf{j} \cdot \mathbf{v} \, d\Omega,
$$

The surface integral may be simplified by recalling that $\partial \Omega^em = \partial \Omega^em_h \cup \partial \Omega^em_b$. On $\partial \Omega^em_h$, the natural BC (4.16) imposes the nullity of the integrand, and thus of the relevant contribution to the surface integral as well. On $\partial \Omega^em_b$, the test function $\mathbf{v}$ is parallel to $\mathbf{n}$ for the space characterizing property $\mathbf{n} \times \mathbf{v}|_{\partial \Omega^em} = 0$ (see the remark on electromagnetic BC in §4.2.3). Therefore $(\mathbf{n} \times \mu^{-1} \nabla \times \mathbf{A}) \cdot \mathbf{v}$ is identically null on $\partial \Omega^em_b$, for a vector product is always orthogonal to its factors. Then the nullity of the integrand implies the nullity of the relevant contribution to the surface integral as well.

For the above remarks on BC, (4.75) simply reduces to

$$
\int_{\Omega^em} \frac{1}{\mu} (\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{v}) \, d\Omega = \int_{\Omega^em} \mathbf{j} \cdot \mathbf{v} \, d\Omega.
$$

(4.76)

For the sake of compactness, the symmetric, positive definite, bilinear form $L^{ms} : (H^1(\Omega^em))^3 \times (H^1(\Omega^em))^3 \to \mathbb{R}$ may be introduced, defined as

$$
L^{ms}(\mathbf{u}, \mathbf{v}) := \int_{\Omega^em} \frac{1}{\mu} (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \, d\Omega,
$$

(4.77)

together with the linear functional $f^{ms} : (H^1(\Omega^em))^3 \to \mathbb{R}$, defined as

$$
f^{ms}(\mathbf{v}) := \int_{\Omega^em} \mathbf{j} \cdot \mathbf{v} \, d\Omega.
$$

(4.78)

This allows rewriting the weak form as the problem to seek $\mathbf{A} \in \mathbf{V}^{ms}$ such that

$$
L^{ms}(\mathbf{A}, \mathbf{v}) = f^{ms}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}^{ms}.
$$

(4.79)
We now need to define the finite dimensional approximating subspace (4.28). To this aim, in addition to the set \( \mathbb{P}_r \) of polynomials of total degree at most \( r \), defined by (4.66), let us introduce the space \( \tilde{\mathbb{P}}_r \) of homogeneous polynomials of degree \( r \). Based on the above ingredients, the set of polynomials

\[
\mathbf{R}_r := (\mathbb{P}_{r-1})^3 \oplus \left\{ \mathbf{p} \in (\tilde{\mathbb{P}}_r)^3 \mid \mathbf{p} \cdot \mathbf{x} = 0 \right\}
\]

is defined. Then, one chooses the approximating subspace to be a set of element wise polynomials. Two possibilities are usually available, namely the space of Nédélec elements of the first type \[107\]

\[
\mathbf{V}_{n,1}^{ms} := \{ \mathbf{v} \in H^0(\text{curl}; \Omega_{em}) \mid \mathbf{v}|_{\Omega_i} \in \mathbf{R}_r \ \forall \Omega_i \in \mathcal{C}(\mathcal{G}) \}
\]

and the space of Nédélec elements of the second type \[108\]

\[
\mathbf{V}_{n,2}^{ms} := \{ \mathbf{v} \in H^0(\text{curl}; \Omega_{em}) \mid \mathbf{v}|_{\Omega_i} \in (\tilde{\mathbb{P}}_r)^3 \ \forall \Omega_i \in \mathcal{C}(\mathcal{G}) \}.
\]

The geometrical interpretation of the above spaces and the definition of the spanning basis functions is similar to the case of electrostatics, the fundamental difference being that edges and not nodes are concerned. Particularly, \( n = |\mathcal{E}(\mathcal{G})| \), that is, the number of edges in the computational grid, is the index of the approximation. The degrees of freedom are associated with the edges, and particularly they consist of the scalar product of the interpolated vector field (i.e., vector potential) with the edge direction. For this reason, Nédélec elements are also termed edge elements and the stemming version of the FEM is called edge finite element method. Another denomination is mixed elements and, finally, Whitney elements, for they were originally defined and used by Whitney to compute Betti numbers of smooth manifolds \[166\].

Edge based, instead of node based, vector basis functions \( \varphi_i \) are used to span the discrete, approximating spaces (4.81) or (4.82), which we shall simply term \( \mathbf{V}_n^{ms} \) from now on. Then, the reasoning is similar to what already illustrated for electrostatics and the approximate solution \( \mathbf{A}_n \in \mathbf{V}_n^{ms} \) is thus expressed in the form (4.29) as

\[
\mathbf{A}_n(\mathbf{x},t) = \sum_{j=1}^{n} A_j(t) \varphi_j(\mathbf{x}).
\]

For what above, the sum \( \Sigma \) of the edge degrees of freedom over any closed loop \( \gamma \), and particularly over the border of the facets of edge elements, is the sum of scalar products of the interpolated field, i.e., vector potential \( \mathbf{A} \). This is exactly the circulation of the vector potential over a closed path, and, for the curl theorem (A.23), equals the integral of the curl of \( \mathbf{A} \) over any surface having such a path \( \gamma \) as border. In general, the above surface consists of a patch of facets. If we assume \( \gamma \) be the border of a single facet, then \( \Sigma \) equals
the integral of \( B = \nabla \times A \) on such a facet. Therefore, any two edge elements neighboring for a facet see the same magnetic flux through the common facet, by construction. For this reason, edge elements are termed \( B \) conformal, or, more generally, \textit{curl conformal} (with reference to the fact that the curl of the interpolated field is conserved through the facets. Owing to this property, otherwise detrimentally appearing spurious (meaning, non physical) modes are automatically eliminated from the numerical solution.

After this said, from the practical point of view we can define the matrix and vectors

\[
L^{ms} := [L^{ms}(\varphi_i, \varphi_j)], \quad u^{ms} := [A_i], \quad f^{ms} := [f^{ms}(\varphi_i)],
\]

and test the weak form (4.79) with test functions \( \varphi_i \). With straightforward passages, basically exploiting the bi-linearity of the form \( L^{ms} \), the system

\[
L^{ms} u^{ms} = f^{ms}
\]

is obtained, consisting of \( n \) equations in the \( n \) unknowns \( A_i \). The removal of the nullspace of the differential operator is not accomplished by means of the Coulomb gauge or similar. Rather, from the numerical standpoint it is more practical to use a spanning tree technique to gauge the vector potential, so as to guarantee its uniqueness [2].

Also in the case of magnetostatics, time is not appearing with a partial derivative and system (4.85) is simply solved at any time step with updated values of the current density field \( j \). Formally, the system (4.85) is written as

\[
L^{ms}_{k,j} u^{ms}_{k,j} = f^{ms}_{k,j},
\]

where the subscript \( k \) refers to the number of the time step to solve. Furthermore, due to the solution dependent properties of ferromagnetic inclusions (the splitter plates), the magnetostatic problem is nonlinear, with \( L^{ms}_{k,j} = L^{ms}_{k,j}(u^{ms}_{k,j}) \), and Newton-Raphson iterations are required for its solution. This introduces the additional subscript \( j \), referring to the Newton-Raphson iteration to solve.

It is simple to solve both (4.71) and (4.85), or rather the systems stemming thereafter with Newton-Raphson method. This is because from the symmetry and positive definiteness of the bilinear forms \( L^{es} \) and \( L^{ms} \), respectively, symmetric and positive definite matrices \( L^{es} \) and \( L^{ms} \), respectively, are originated. As a consequence, the numerical solution may use Cholesky factorization, in case of a direct method, or, which is particularly suited for large scale problems, the preconditioned conjugate gradient method, in case of an iterative method. Domain decomposition techniques may be adopted to build quasi-optimal preconditioners, and naturally exploiting parallel computing [122, 123, 142, 152].
A different mesh is used, in general, for the electromagnetic problem and the fluid dynamic problem. Also, the electromagnetic problem is usually solved less frequently than the fluid dynamic one, say once every $5\mu s$, and the relevant electromagnetic quantities involved in the fluid dynamic problem are assumed not to change meanwhile.

### 4.3.6 Discretization of the Electric Network Problem

The governing equation (4.18) of the electric network is frequently turned into a linear, second order ordinary differential equation (ODE) by differentiating once with reference to time. Our solution strategy is different, and we keep the integro-differential structure. We have implemented the procedure by coding a suitable user-defined boundary condition. We keep the time discretization scheme arising from the time discretization of the fluid dynamic problem. In what follows we will provide a time discretized approximation of the relevant quantities occurring in equation (4.18). The same notation and assumptions described in (§4.3.3) hold.

The generator voltage supply is simply evaluated at time $t_{k+1}$, yielding

$$u_{G,k+1} = V_{DC} + V_{AC} \sin(\omega t_{k+1} + \theta). \quad (4.87)$$

The electric charge removed from the capacitor banks amounts to an integral, which is immediately approximated by the trapezoidal rule. It is convenient to update the value obtained at the previous time step, yielding

$$q_{k+1} \approx q_k + \frac{i_{k+1} + i_k}{2} \Delta t_k. \quad (4.88)$$

Then, the capacitive voltage drop is immediately written as

$$u_{C,k+1} = u_{C,0} - \frac{q_{k+1}}{C}. \quad (4.89)$$

A straightforward evaluation is also applicable to the ohmic voltage drop at the resistor, yielding

$$u_{R,k+1} = R i_{k+1}. \quad (4.90)$$

The inductive voltage drops involves a derivative, which can be approximated by means of the difference quotient. If one chooses a forward difference, then

$$u_{L,k+1} \approx L \frac{i_{k+1} - i_k}{\Delta t_k} \quad (4.91)$$

is obtained. Finally, the arc voltage drop is approximated by means of the value available from the previous time step, namely $u_k$. 
4.3. PROBLEM SOLUTION

By plugging the voltage contributions (4.87), (4.89), (4.90), (4.91), some of which are approximated, and the approximated expression (4.88) into (4.18), an algebraic expression is obtained in the unknown variable \( i_{k+1} \), that is,

\[
 u_{G,k+1} + u_{C,0} - \frac{q_k}{C} - \frac{i_{k+1} - i_k}{2C} \Delta t_k = Ri_{k+1} + L \frac{i_{k+1} - i_k}{\Delta t_k} + u_k. \tag{4.92}
\]

Since the equation is of first degree, it is immediate to solve it, yielding the updated value of the current at time \( t_{k+1} \), which reads

\[
 i_{k+1} = \frac{u_{G,k+1} + u_{C,0} - \frac{q_k}{C} - u_k + \left( \frac{L}{\Delta t_k} - \frac{\Delta t_k}{2C} \right) i_k}{R + \frac{L}{\Delta t_k} + \frac{\Delta t_k}{2C}}. \tag{4.93}
\]

4.3.7 Data Interpolation

The many differential problems at hand are solved independently and with different computational grids. Any grid is associated with a cloud of notable points, where the physical quantities are sampled. For instance, cells centers in a FVM mesh and cell vertices in a node based FEM mesh. Different grids are associated with different point clouds. Then the need arises to pass data from one problem to the other, in such a way to overcome the difference in the point clouds. The general procedure implemented is hereafter outlined.

A collection, or a “cloud”, of source points \( \{x_i\}_{i \in I} \) is given, where \( I \) is a suitable index set. A given quantity \( \varphi \) is defined over such a collection, \( \varphi_i \) being the value in \( x_i \). We want to find an estimate of the value \( \overline{\varphi} \) of the quantity \( \varphi \) in a target point \( \overline{x} \). If a source point falls close enough to the target point, then the value of such a source point is used for the target point (a very small proximity threshold is defined, and the closest source point under tolerance is chosen). This way, the possible case of a target point actually coinciding with some source point is handled exactly.

Otherwise, the strategy adopted is by means of a distance weighted interpolation scheme. First, a sphere \( S \) is defined, centered in \( \overline{x} \) and with radius \( R \). Then the subset \( \overline{I} \subseteq I \) of the at most \( N \) closest source points to \( \overline{x} \) is found and, if such a subset is not empty, the interpolated value is defined as

\[
 \overline{\varphi} := \frac{\sum_{i \in \overline{I}} \varphi_i/r_i}{\sum_{i \in \overline{I}} 1/r_i}, \tag{4.94}
\]

where \( r_i = \|r_i\| = \|\overline{x} - x_i\| \) is the distance of the target point \( \overline{x} \) from the \( i \)-th source point \( x_i \); see Figure 4.6, where \( N = 5 \). In case \( \overline{I} = \emptyset \), then the radius \( R \) has to be enlarged. The radius \( R \) is calibrated over the average grain of the
cloud of source points and can be modified locally, to account for coarser or finer density regions.

Finally, since a given differential problem may consist of more sub-domains and since the data defined over some sub-domain may be not of interest when data passing is concerned, then some “domain-pass” filtering is needed. This is very easily accomplished by associating points in a cloud to geometrical domains.

4.4 Results

The computational approach outlined in the previous sections has been tested on some reference cases. Even though all of our models are more or less idealized imitations of reality, we tried to reproduce the essential features that can be encountered in low voltage circuit breakers. The simulation activity is still in a preliminary stage and some time will be needed to reach the level of an industrial tool to be used in day-by-day engineering practice. In what follows we report a very short account, in order to be aware of the potentialities of the method. A comprehensive description and validation is outside the scope of this thesis.

4.4.1 Simple Test Cases

A first case is constituted by two parallel conductive rails, with or without a central ferromagnetic splitter plate. The two rails are plugged into the electric
network described in §4.2.4, where a discharging capacitor provides the voltage supply. The voltage generator has not been used. Many experimental tests have been carried out on such a case at ABB Corporate Research Center, in the framework of an activity which is not part of our work. Figure 4.7 shows a comparison between an experiment and a numerical simulation, in the same testing conditions. The experimental frames have been filmed by means of a fast camera. The light pattern is shown in the experimental sequence, whilst the current density field is shown in the simulated sequence.

A general agreement, at a qualitative even though not quantitative level can be appreciated from the comparison. More or less obviously, the real shape and size of the arc cannot be exactly compared, since the light pattern and current density are related but in a non trivial manner. Furthermore, features at such level of detail vary from one experiment to the other, even in nominally

Figure 4.7: A sequence of frames showing a comparison between an experiment and the relevant magnetohydrodynamic simulation of a simple case. The (filtered) level of light intensity is shown in the experiment, while the current density field is shown in the simulation.
identical conditions. Nonetheless, the main trend of arc evolution appears to have been reasonably simulated. Particularly, the transition throughout the central region, where the splitter plate is located, is characterized by a non-symmetrical motion of the two arc branches, both in the experiment and in the simulation, with the rightmost branch going forward along the splitter plate and the leftmost staying back at the front of the splitter plate. This can be easily explained by the opposite action of Lorentz forces acting upon the two branches, once symmetry has been broken. Also, as soon as the rightmost branch reaches the rear of the splitter plate and reconnects with the left rail, then the leftmost branch is short-circuited both in the experiment and in the simulation.

A multi-core workstation, currently of large availability in the framework of industrial research and product development offices, has been used to carry out the computations. A single computation of such a level of complexity is run overnight.

4.4.2 Industrial Test Cases

The computational MHD approach has also been applied to an industrial test case. The circuit breaker under test is the idealized, yet rather realistic model of MCB by ABB already shown in Figures 4.1 to 4.5. Also this case is addressed by means of a multi-core workstation, currently of large availability in the framework of industrial research and product development offices. Three cores have been used to execute a parallel computation. A single computation of such a level of complexity is run in approximately one day.

The circuit breaker is subject to a simulated single pole short-circuit test. A 50 Hz voltage generator supplies 230 V (r.m.s.), corresponding to a voltage peak approximately 360 V high. The prospective current is equal to 3000 A (r.m.s.). Apart from the circuit breaker under test, the test network also includes a 0.107 mH inductor and a 70 mΩ resistor, so that the impedance is equal to 77 mΩ and \( \cos \varphi = 0.9 \). The insertion angle is 30 deg.

The air plasma temperature field is shown in Figure 4.8. Precisely, a cross section with the mid plane of the breaker is displayed. A “reversed” gray scale color map is used, and dark black spots are hotter than light white ones. The temperature range has been arbitrarily fixed to be 10 000 – 28 000 K. The time evolution is summarized into eight, selected frames ordered left to right and then top down. The duration of the simulated phenomenon is approximately 2.7 ms and the time of each frame is reported on the top part of each picture. The selected frames are representative of notable instants during the arc evolution. The arc migration from its ignition place (top right in each frame) to
Figure 4.8: Air plasma temperature field in an industrial test case.
Figure 4.9: Current density field in an industrial test case.
the extinguishing chamber (center of each frame) is clearly recognizable. With reference to the conducting path shown in each frame and owing to the polarity of voltage assigned to the circuit breaker during the test, each cathode always stands above the subsequent anode. The cathodes are observed to be hotter than the anodes, as expected from the theory outlined in §3.

The current density field is shown in Figure 4.9. The mid plane is used again to produce a cross section of the circuit breaker, and the magnitude of the vector field is reported (higher values correspond to dark spots, according once again to a “reversed” gray scale color map). The range of current density values has been arbitrarily fixed to be $0 - 10^8 \, A/m^2$. The very same eight, selected frames are used, as in the case of the temperature field. Similar conclusions as above about the arc evolution can be drawn also in the case of current density.

The arc voltage time history is shown in Figure 4.10. The numerical, MHD solution is displayed by a thick line. For the sake of comparison, a reference experimental oscillogram is also displayed (thin line). Actually, the real oscillogram is representative of a real short circuit test, starting with the opening of the electrical contacts of the circuit breaker, while in the simulated case the arc is ignited by modeling the burning of a thin metal wire originally connecting
the two electrodes. As a consequence, the starting phases of the resulting arc voltages in the two cases look rather different. Nonetheless, the simulated arc seem to recover a similar behavior than the real one, by producing a voltage build up similar to the one observed in the test lab oscillogram.

Particularly, the maximum amplitude of the voltage drop is of the order of $250 \text{ V}$, which is in line with the theoretical predictions based on the presence of eight splitter plates in the extinguishing chamber (i.e., 8+1 cathodic and anodic voltage drops, approximately $20 \text{ V}$ each, plus the voltage drop due to the arc column, which can be roughly estimated to be of the order of some tens of $V$ by considering a conducting, $0.02 \text{ m}$ high cylinder with $0.004 \text{ m}$ radius and the maximal air plasma electrical conductivity $10^4 \text{ S/m}$; see Figure 3.10). Also the time needed for the arc to reach and entirely enter the extinguishing chamber, thus producing the maximum voltage drop, is observed to be of the order of $2 \text{ ms}$, both in the real and simulated case. It is probably not useless to point out that this result has been obtained by physically describing physical features (electrical conductivity and other physical properties of the medium, radiation with a participating medium, arc roots, etc.) starting from rather prime principles, and not by fine tuning the MHD model so to fit the experimental behavior.

From the inspection of the frames reported in Figures 4.8 and 4.9, one may get a deeper insight into the oscillogram of Figure 4.10.

4.4.3 A Posteriori Evaluation of the Magnetic Reynolds Number

In §3.7.6, the dimensionless magnetic Reynolds number (3.122), or Lundquist number

$$Rm := \frac{|\nabla \times (\mathbf{u} \times \mathbf{B})|}{|\eta \nabla^2 \mathbf{B}|},$$

has been introduced, explaining its role, when it is much less than unity, in allowing the decoupling of Navier-Stokes equation from Maxwell equations when solving the magnetohydrodynamic problem. Furthermore, based on the estimate (3.123), we expect the magnetic Reynolds number to be low for typical plasma conditions which are reasonably met in an arcing LV circuit breaker. Particularly, values approximately below the order of some hundredths are expected.

The estimate is confirmed by results of simulations carried out on test cases close to low voltage arc conditions, showing numerical evidence that the magnetic Reynolds number is well below the critical unitary threshold; see §4.4.3. For instance, Figure 4.11 shows maps of magnetic Reynolds number in the case
Figure 4.11: Magnetic Reynolds number maps in a simple test case close to LV arc plasma conditions.
of an arc traveling through a square duct with one ferromagnetic splitter plate in the middle, a case very similar to a low voltage, miniature circuit breaker. Three instants are depicted, namely before, when entering and in the middle of the splitter plate. In any case the magnetic Reynolds number is always lower than $1/100$ and usually it is even lower than $1/1000$.

In the technical literature of low voltage circuit breaker simulation this fundamental fact is frequently unmentioned, to the authors knowledge. Of course what we have shown here is by no means a mathematical proof that the problem we solve is well posed, and whatever number of results \textit{a posteriori} cannot prove such a thing. Nonetheless, there is a sound justification that the mathematical framework is reasonable and self coherent.

### 4.5 Conclusions

The current chapter reports the computational approach to the magnetohydrodynamics equations governing the arc plasma, applied to the case of a low voltage circuit breaker.

The level of approximation of plasma modeling, along with the corresponding set of equations to be solved, has been outlined in the theoretical review in chapter §3. Particularly, the adoption of the MHD description, according to which the plasma is modeled as a single fluid continuum, is justified when the local thermal equilibrium holds. This implicitly implies that the scope of the simulations is the macroscopic flow.

All computations have been carried out by means of the weak coupling of different numerical tools. Particularly, we have used the commercial finite volume code Fluent for Navier-Stokes equations, governing the flow, and for the radiative transfer equation. Then we have used the commercial code Oofelie for Maxwell equations, with a curl conform, edge finite element approach for the equations of magnetostatics and a traditional, node based finite element approach for the equations of electrostatics. Data passing, the lumped parameter model of the electrical network, the arc root model, material ablation and the nonlinear characterization of the physical properties have been addressed by means of an original corpus of user subroutines. The simple cases considered have shown an acceptable agreement with the available experimental results. The plasma flow is \textit{a posteriori} found to be in the low magnetic Reynolds regime, i.e., not in contrast with the weak coupling of Maxwell and Navier-Stokes equations.
Chapter 5

Black Box Arc Models

5.1 Summary

Black box arc models are dealt with in this chapter, in order to develop a fast, though extremely simplified, method for describing the behavior of low voltage circuit breakers inside real networks. The arcing circuit breaker is described as a nonlinear resistive component, whose conductance is governed by an ordinary differential equation (ODE).

The arc-network interaction problem is first introduced and set in the framework of differential algebraic equations (DAE). The case of a RLC network coupled with simple black box models is illustrated in some detail, showing a trivial, closed form reduction to a system of ODE. The general case is addressed by the classical techniques of index reduction [43, 42, 6, 115, 87] or of substitute equations [37], and solved by means of the stiff problem solvers that can be found in the Matlab/Simulink commercial code [136, 137].

The black box models currently available for high-voltage circuit breakers are introduced and reviewed, starting from the classical works from Mayr [88, 89], Cassie [21] and Schwarz [135]. Such models characterize the time evolution of arc conductance as a merely thermal balance problem. The so-called field correction, first introduced by Rieder and Urbanek [125] with reference to non equilibrium theory, allows accounting for non equilibrium phenomena which are expected to strongly affect the low current regime of an arc close to extinction. Based on such considerations, a black box arc model is proposed for low voltage
circuit breakers. The model is of Schwarz type in the high current regime, while in the low current regime accounts for non equilibrium effects in the cold plasma by means of the field correction.

The problem of model parameter identification is addressed and a mathematical algorithm is proposed, solving a constrained optimization problem by suitably coupling gradient moves with heuristic search methods. Black box modeling is applied to industrial low voltage circuit breakers and experimental results are compared with the synthetic behavior stemming from the model, with best fit value of its parameters. A strikingly good descriptive capability of the proposed model is observed, with a remarkable improvement with reference to equilibrium theory based models. Unfortunately, the large scattering in between nominally identical tests and breakers turns out into a large scattering of model parameters, so that the model can hardly be envisaged to be predictive in different conditions than those used for its calibration. Nonetheless, the experience gathered seems to confirm and validate the interesting theoretical result of the relevance of non equilibrium phenomena in current zero physics and circuit interruption.

5.2 Arc-Network Interaction

Frequently, in the technical applications, the interaction between an electric network and a circuit breaker in the switching arc phase has to be investigated. In most of the cases, the computational burden of the mathematical model describing the arc must be extremely reduced. When this happens, multiphysical models are inadequate and the only possibility is to rely on simplified models, usually accounting for electrical, macroscopic properties only, such as electric current and voltage time histories or other electrical quantities thereafter deduced, such as conductance time history.

Black box arc models are hence defined when the electrical behavior is of importance, rather than internal physical processes [111, 112]. The expression black box wants to express the feature of such models of being integral, that is, they describe the electric arc as a lumped parameter element, where the circuit breaker geometry and the physical properties of materials and gases are implicitly summarized into a given, finite set of parameters. Such parameters need being identified by means of a reference experimental test case, typically a short-circuit test, and hopefully retain the physical characteristics of the circuit breaker at hand. In case the black box model has an underlying, solid physical base, then the hope is reasonable that the fitted value of parameters remain valid, at least to some extent, in different conditions than those used for their determination.
We will start our exploration of the multitude of available black box models for electric arcs from a group whose members we call \textit{classical models}. The attribute “classical” both stands for the historical context, since such models where the first to be proposed, and for a deeper physical concept. Precisely, the classical local thermal equilibrium (LTE) hypothesis is implicitly assumed, which, as will be shown in §5.4, prevents from accounting for dielectric phenomena adequately and limits such models to purely thermal phenomena. The fact will be seen in §5.6 to have relevance when an accurate description of the current zero zone is required. In the case of low voltage arcs, we will propose to use a classical model only to approach the current zero, and later on to use a more sophisticated non equilibrium model.

5.2.1 Problem Setting

A mathematical description of the usage of black box models combined with an external network description is given by the differential system

\[
\begin{align*}
    i(t) &= g(t) \cdot u(t) \\
    \dot{g}(t) &= \varphi(t, g(t), u(t), \dot{u}(t), i(t); \theta) \\
    \eta(t, \mathbf{u}(t), \ldots, \mathbf{u}^{(p)}(t), \mathbf{u}(t), \mathbf{i}(t), \ldots, \mathbf{i}^{(p)}(t), i(t)) &= 0,
\end{align*}
\]

which we now examine in greater detail.

A first characterizing property of all black box models is the assumption that the arc has a resistive behavior. The current-voltage characteristic of the arc may be thus ruled by Ohm’s (first) law, albeit of a nonlinear kind. In terms of conductance, Ohm’s law reads

\[ i = gu, \]

where \( i \) is the arc current and \( u \) the voltage drop across the arc, or the arc voltage. Ohm’s law is the algebraic constraint appearing first in (5.1).

Arc conductance \( g \) is assumed to be non constant and related to the arc internal energy content. From the mathematical point of view, arc conductance is assumed to be ruled by a first order, nonlinear ordinary differential equation (ODE), which, in a rather wide variety of cases, we may assume to include current, voltage and voltage time derivative, as well as conductance, as appearing in the second of (5.1). The function \( \varphi \) depends on the particular black box model employed, as shown in next sections, and may leave some variables out of consideration, as a special case. Additional electrical quantities, or of different physical nature, could be introduced, from the theoretical viewpoint. The
vector $\theta$ collects model parameters, in a number depending on the particular model employed.

The third equation in (5.1) is, in the general case, a set of integro-differential equations (possibly purely algebraic or purely differential), in vector form, describing the behavior of the electric network in which the circuit breaker is inserted. As such, the vector function $\eta$, describing the integro-differential Kirchhoff equations and component voltage-current characteristics, includes voltages and currents, together with their time derivatives up to a certain order $p \in \mathbb{N}_0$ (not necessarily all derivatives have to be present). Arc voltage $u$ and arc current $i$ have been explicitly evidenced and placed out of the vectors $u$ and $i$, which collect voltage drops and currents, respectively, across other components of the rest of the network. In the very typical case of RLC networks (see §5.2.3), $\eta$ can be trivially reduced to algebraic equations and a single, linear, second order ODE, and with constant coefficients. Function $\eta$ may be nonlinear, e.g., if diodes, variable resistances, variable inductances, variable capacitances or any other nonlinear element is present in the network.

Time $t$ has been explicitly indicated in the list of input arguments of $\eta$, so to include also the case when the structure of the network - and thus of the governing set of equations - may change from some instant on (consider, e.g., switching of a three phase network). The usual handling of such kind of discontinuities in the vector function $\eta$ is by means of switching functions. When the sign of the switching function changes, a discontinuity occurs. Integration then stops, and is restarted again after the discontinuity, with the network conditions at the end of the previous period used as initial conditions for the new period. Numerical methods for the treatment of discontinuities are described in [36].

When simulating the arc-network interaction, the arc model has been identified, so that vector $\theta$ is known. During model identification, such vector is the unknown of the problem, but the rest of the network is usually substituted (always in the case of this thesis) by experimentally measured arc voltage and arc current time histories. Consequently, problem (5.3) is replaced by

$$
\begin{cases}
    i(t) = g(t) \cdot u(t) \\
    \dot{g}(t) = \varphi(t, g(t), u(t), \dot{u}(t), i(t); \theta) \\
    u(t) = \overline{u}(t) \\
    i(t) = \overline{i}(t),
\end{cases}
$$

where $\overline{u}(t)$ and $\overline{i}(t)$ are known functions of time, describing the experimental signals. We will always use suitably post-processed signals, as described in
5.2. ARC-NETWORK INTERACTION

§2.3. Also the time derivative of electrical variables, formally appearing in the first equation of (5.3), are actually provided by the Savitsky-Golay filter.

When solving problem (5.1), the algebraic constraint (5.2) may always be used to eliminate arc current or arc voltage. From the theoretical standpoint, the operation has no consequence in exact mathematics but could introduce some difference when a numerical method is used and finite arithmetic, such as double precision arithmetic, is necessarily introduced. Except for pathologic cases, the effects are harmless, provided that a numerically stable method is adopted. Contrarily, when model identification is concerned, then the choice to eliminate \( u \) or \( i \) could lead to different solutions, because a nonlinear optimization problem must be solved (see §5.7). In this case it is not possible to predict which action is best but by \textit{a posteriori} comparison of results. We have chosen to eliminate arc current from Ohm’s law, drop the latter and write the rest of (5.3) in terms of arc voltage only.

5.2.2 Problem Solution

Usually, integro-differential equations are cast into purely differential equations, by further differentiating until integrals are completely removed. In the general case, the system (5.1) can only be converted into a set of differential algebraic equations (DAE), for not all algebraic constraints may be explicitly eliminated (or it could be more convenient not to do so). Therefore, problem (5.1) may be rewritten as

\[
f(\dot{x}(t), x(t), y(t), t) = 0,
\]

where \( x \in \mathbb{R}^n \) is a vector collecting differential variables and \( y \in \mathbb{R}^m \) is a vector collecting algebraic variables. Time \( t \in \mathbb{R} \) is the independent variable and

\[
f: \mathbb{R}^{2n+m+1} \rightarrow \mathbb{R}^{n+m}
\]

is a function describing \( n \) differential equations and \( m \) algebraic constraints. By the standard argument of introducing auxiliary variables, time derivatives of higher orders may be removed and first order derivatives may only be accounted for, which explains the scripture of (5.4). Also, differential systems up to the first order can be solved by the majority of the numerical methods.

It may happen that, after successively time differentiating \( m \) times the DAE
(5.4), the resulting system of equations, i.e.,
\[
\begin{align*}
  f(\dot{x}(t), x(t), y(t), t) &= 0 \\
  \frac{d}{dt} f(\dot{x}(t), x(t), y(t), t) &= 0 \\
  &\vdots \\
  \frac{d^m}{dt^m} f(\dot{x}(t), x(t), y(t), t) &= 0,
\end{align*}
\]
can be cast, by algebraic manipulations, into the set of ODE
\[
[\dot{x}(t), \dot{y}(t)]^T = \tilde{g}(x(t), y(t), t).
\]
If \(m\) is the smallest integer such that this holds, then the DAE (5.4) is said to have differential index \(m\) [42]. Usually, the higher the index, the more difficult the numerical solution to the DAE. In many simple but important cases like, e.g., linear networks with constant parameters and simple arc models (like the one we propose for low voltage arcs), all algebraic variables \(y(t)\) may be eliminated by means of the algebraic constraints (see §5.2.3), and the set of DAE natively reduces to the set of ODE
\[
\dot{x}(t) = g(x(t), t),
\]
where
\[
g : \mathbb{R}^{n+1} \to \mathbb{R}^n
\]
is a function describing \(n\) differential equations. By the previous definition, ODE have differential index 0.

System (5.1) must be complemented with a suitable set of initial conditions (IC). Usually, a Cauchy initial value problem is formulated. In the general case of a system of DAE, the initial values \(\dot{x}_0, x_0\) and \(y_0\) at time \(t = t_0\) of \(\dot{x}(t), x(t)\) and \(y(t)\), respectively, must satisfy
\[
f(\dot{x}_0, x_0, y_0, t_0) = 0,
\]
and so they cannot be specified independently. As an example, initial arc conductance, voltage and current must be such that \(i_0 = g_0 u_0\). The number of degrees of freedom of a system of DAE is defined as the number of the \(2n + m\) initial conditions that can be specified independently. Frequently the number of degrees of freedom coincides with \(n\). This is always the (trivial) case when a system of ODE (5.5) is governing problem (5.1) and only \(x_0\) is to be provided. In the general case of higher index DAE, constraints hidden in the set (5.4) can reduce the number of degrees of freedom below \(n\), meaning that there are dependencies among differential equations that can become apparent after differentiation and algebraic manipulations [37].
The scope of this thesis is restricted to testing lab circuits, which we have modeled as linear, constant parameter, RLC systems. Fairly general networks, e.g., including electronic components, or even simple networks with particularly complicated arc models, e.g., including physical variables of other nature than electrical, may fall in the category of high index DAE. This is a well-known issue in modeling physical systems and unfortunately it is rather difficult. The usual technique to solve such higher index DAE is by index reduction [43, 42, 6, 115, 87], i.e., through differentiation and algebraic manipulations, the index is lowered to 0 or 1. An alternative approach is that of substitute equations [37], allowing index reduction and consistent initialization of higher index DAE without changing the original equations. The latter property is a valuable benefit for the modeler, who retains full control over the physical problem.

The system (5.1) may be numerically solved by means of one of the many methods suitable for differential problems. If only ODE have to be tackled, then the problem is trivial and virtually any classical method is applicable (Runge-Kutta, Predictor Corrector methods, BDF methods or even the elementary explicit Euler’s method, etc.). In this case, the problem is only how many resources need to be allocated to reach a given accuracy. If nonlinear electrical networks, including electronic components, must be coupled with a low voltage circuit breaker model, then the general case of DAE must be solved and a stiff method must be applied. We employed Matlab-Simulink, which offers, among others, a modified Rosenbrock formula of order 2 [136] or a multistep, variable order method based on the numerical differentiation formulas (NDF) [136, 137].

### 5.2.3 The RLC Network Case

In order to exemplify in a concrete case the concepts and methods exposed in the previous sections, we now consider a RLC network hosting a low voltage circuit breaker in arcing phase. Initially we let the arc model be generic and shall eventually see how to handle the two cases required in our proposed model for low voltage arcs.

We refer to the network shown in Figure 5.1. The governing system of DAE
(5.3) may be written as

\[
\begin{align*}
  i(t) &= g(t) \cdot u(t) \\
  \dot{g}(t) &= \varphi(t, g(t), u(t), \dot{u}(t), i(t); \theta) \\
  u_1(t) &= L \cdot \frac{di}{dt} \\
  u_2(t) &= R \cdot i(t) \\
  u_3(t) &= \frac{1}{C} \left( q_0 + \int_{t_0}^{t} i(t') \, dt' \right) \\
  u_1(t) + u_2(t) + u_3(t) + u(t) &= 0.
\end{align*}
\]

The vector function \( \eta \) therefore reads

\[
\eta = \begin{bmatrix}
  u_1 - L \frac{di}{dt} \\
  u_2 - Ri \\
  u_3 - \frac{q_0}{C} - \frac{1}{C} \int_{t_0}^{t} i(t') \, dt' \\
  u_1 + u_2 + u_3 + u
\end{bmatrix}.
\]

The third entry of \( \eta \) yields an integral equation (and plugging the various \( u_i, i \in \{1, 2, 3\} \), into the fourth entry of \( \eta \) an integro-differential equation is obtained). In §5.2.2 we mentioned the possibility to turn integro-differential equations into purely differential equations by further differentiating. In the

\[
\begin{array}{cccc}
  1 & 2 & 3 & 4 \\
  \text{L} & \text{R} & \text{C} & \text{\textbullet}
\end{array}
\]

Figure 5.1: Schematic of a RLC network: 1 - inductor with inductance \( L \); 2 - resistor with resistance \( R \); 3 - capacitor with capacitance \( C \) and initial charge \( q_0 \) on its banks; 4 - low voltage circuit breaker in arcing phase.
case at hand, one immediately gets
\[
\frac{du_3}{dt} = \frac{i(t)}{C}.
\] (5.9)

The form (5.4) is obtained after defining the vector of differential variables, which in the case at hand reads
\[
\mathbf{x}(t) = [i(t), g(t), u_3(t)]^T,
\] (5.10)
the vector of algebraic variables, which in the case at hand reads
\[
\mathbf{y}(t) = [u_1(t), u_2(t), u(t)]^T,
\] (5.11)
and the vector function
\[
\mathbf{f} = \begin{bmatrix}
i - gu \\
\frac{dg}{dt} - \varphi(t, g, u, \frac{du}{dt}, i; \theta) \\
u_1 - L\frac{di}{dt} \\
u_2 - Ri \\
\frac{du_3}{dt} - \frac{i}{C} \\
u_1 + u_2 + u_3 + u
\end{bmatrix}.
\] (5.12)

The case considered is simple enough to be easily turned from a system of DAE into a system of ODE. It must be noted, anyway, that this extremely special and fortunate condition is hard to hold in the general case, and offers huge benefits from the mathematical standpoint at the price of a reduced manipulation effort. As indicated in §5.2.2, the idea is to differentiate \(\mathbf{f}\) and to rearrange algebraically. In the case at hand, we can time differentiate Kirchhoff law for voltage and plug in the contributions of the various network components, obtaining
\[
\begin{cases}
i = gu \\
\frac{dg}{dt} = \varphi(t, g, u, \frac{du}{dt}, i; \theta) \\
L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i + \frac{du}{dt} = 0.
\end{cases}
\] (5.13)

We prefer down scaling differential equations to first order by introducing auxiliary variables, for the reasons explained in §5.2.2. Therefore, to remove the second derivative of current from the third equation we introduce
\[
\xi := \frac{di}{dt}.
\] (5.14)
The first equation is actually an algebraic constraint from which arc voltage can be eliminated elementarily as $u = i/g$. Still we need an expression for the time derivative of arc voltage, which can be obtained in the form

$$\frac{du}{dt} = \left( \frac{di}{dt} + i \frac{dg}{dt} \right) \frac{1}{g^2} \frac{\xi}{g} - \frac{i}{g^2} \frac{dg}{dt}. \tag{5.15}$$

To complete the transformation from DAE to ODE, let us assume that, possibly with the help of (5.15), we can solve the black box arc equation, i.e., the second of (5.13), for the time derivative of conductance in a form like

$$\frac{dg}{dt} = \psi(t, i, g, \xi; \theta). \tag{5.16}$$

Maybe it is not useless to point out that such a fortunate case is not always possible and depends on the particular form of the black box model, for an equation must be solved and the operators must be such to prevent from doing so in closed form. If (5.16) holds, then from (5.13) one immediately gets the system of first order, nonlinear ODE

$$\begin{cases}
\frac{di}{dt} = \xi \\
\frac{dg}{dt} = \psi(t, i, g, \xi; \theta) \\
\frac{d\xi}{dt} = \frac{i}{Lg^2} \psi(t, i, g, \xi; \theta) - \xi \left( R + \frac{1}{g} \right) - \frac{i}{LC}. 
\end{cases} \tag{5.17}$$

The system of ODE (5.17) is the concrete instance of the general vector equation (5.5), with $x = [i, g, \xi]^T$.

We can use (5.17) as a rather general example to illustrate how to solve numerically the arc-network problem. For the sake of simplicity and even though this is not at all a particularly refined choice, we adopt the explicit Euler scheme, according to which derivatives are replaced by finite differences. The step from $t_k$ to $t_{k+1}$ in the time discretized version of (5.17) reads

$$\begin{cases}
i_{k+1} = i_k + \xi_k \Delta t_k \\
g_{k+1} = g_k + \psi_k(\theta) \Delta t_k \\
\xi_{k+1} = \xi_k + \left[ \frac{i_k}{Lg_k^2} \psi_k(\theta) - \xi_k \left( R + \frac{1}{g_k} \right) - \frac{i_k}{LC} \right] \Delta t_k,
\end{cases} \tag{5.18}$$

where quantities with subscript $k$ are relevant to $k$-th time step (and are thus to be considered known when solving the $(k+1)$-th time step), $\psi_k(\theta)$ is shorthand for $\psi(t_k, i_k, g_k, \xi_k; \theta)$ and finally $\Delta t_k$ is the time step duration (which we allow
be varied locally in time, according to some suitable criterion and accounting for the abruptness of evolution of the solution).

The possibility of reaching the form of a system of ODE instead of DAE has huge benefits, including initial condition management in the framework of Cauchy initial value problem, as explained in §5.2.2. In the case at hand, one has to specify the initial current \( i_0 := i(t_0) \) flowing into the network, its initial time derivative \( \xi_0 := (di/dt)_{t=t_0} \) and the initial arc conductance \( g_0 := g(t_0) \). Thanks to the physical sense, one immediately understands that three IC are exactly what required to start the simulation of the problem, for a RLC network is ruled by a second order ODE and the arc initial status clearly adds a third degree of freedom to be fixed. Therefore the number of degrees of freedom of this system of DAE is \( n = 3 \), and \( n = \dim(\text{im}(g)) \), i.e., there are no hidden constraints of differential nature, as expected from the theory (see §5.2.2) in the special case of a system of ODE.

As regards the IC on the initial current slope, which at first may appear difficult to retrieve, one has to go back through the manipulations performed and the ensuing mathematical implications. Typically, switching from an integro-differential equation to a purely differential equation is not without consequences. Precisely, time differentiating the fifth of (5.7), i.e., voltage-current characteristic for the capacitor, and the last of (5.12), i.e., Kirchhoff law for voltage, we removed from the mathematical problem the initial charge \( q_0 \) on the capacitor banks, which is clearly an intrinsically required parameter to define the initial and subsequent status of the capacitor itself. Therefore, one has to go back before such a differentiation and write (the integro-differential) Kirchhoff law for voltage at the initial time \( t_0 \), plug in all of the initial values of the components and find

\[
\left( \frac{di}{dt} \right)_{t=t_0} = -\frac{Ri_0}{L} - \frac{q_0}{LC} - \frac{i_0}{Lg_0}.
\]  

(5.19)

The value of initial current slope deduced from (5.19), together with the other initial values, trivially satisfy the general condition (5.6). Once again, this extreme simplicity is a consequence of the possibility to turn into ODE the system of DAE at hand.

It is of interest to consider the model we propose for low voltage arcs in §5.6. Particularly, two black box equations have to be considered. First the case of Schwarz model (5.36) for the high current regime must be accounted for, which also includes as special cases both the Mayr model and the Cassie model, as illustrated in §5.3.3. Since the black box model is not comprehensive of the \( du/dt \) term, the determination of expression (5.16) is trivial and \( \psi \) coincides
with ϕ. Therefore the system of ODE (5.17) reads

\[
\begin{align*}
\frac{di}{dt} &= \xi \\
\frac{dg}{dt} &= \frac{1}{\tau_0 g^{\alpha-1}} \left( \frac{i^2}{P_0 g^{\beta+1}} - 1 \right) \\
\frac{d\xi}{dt} &= \frac{i}{L \tau_0 g^{\alpha+1}} \left( \frac{i^2}{P_0 g^{\beta+1}} - 1 \right) - \xi \frac{R + \frac{1}{g}}{L} - \frac{i}{LC}.
\end{align*}
\] (5.20)

Then the case of Schwarz-Urbanek model (5.41) for the low current regime must be accounted for. Plugging (5.15) into the black box ODE yields

\[
\frac{dg}{dt} = \frac{1}{\tau_0 g^{\alpha-1}} \left( \frac{i^2}{P_0 g^{\beta+1}} - 1 \right) + \delta \left( \xi - \frac{i \frac{dg}{dt}}{g} \right),
\] (5.21)

which is a linear algebraic equation in \( \frac{dg}{dt} \). The determination of expression (5.16) is thus possible and yet very simple, resulting in

\[
\psi = \frac{dg}{dt} = \frac{1}{g + \delta i} \left[ \frac{1}{\tau_0 g^{\alpha-2}} \left( \frac{i^2}{P_0 g^{\beta+1}} - 1 \right) + g \delta \xi \right].
\] (5.22)

Finally, plugging (5.22) into the system of ODE (5.17) yields

\[
\begin{align*}
\frac{di}{dt} &= \xi \\
\frac{dg}{dt} &= \frac{1}{g + \delta i} \left[ \frac{1}{\tau_0 g^{\alpha-2}} \left( \frac{i^2}{P_0 g^{\beta+1}} - 1 \right) + g \delta \xi \right] \\
\frac{d\xi}{dt} &= \frac{i}{L (g + \delta i)} \left[ \frac{1}{\tau_0 g^{\alpha-2}} \left( \frac{i^2}{P_0 g^{\beta+1}} - 1 \right) + \frac{\delta \xi}{g} \right] - \xi \frac{R + \frac{1}{g}}{L} - \frac{i}{LC}.
\end{align*}
\] (5.23)

The system of ODE (5.20) follows as a special case of (5.23) for \( \delta = 0 \), coherently with the fact that the Schwarz-Urbanek model (5.41) reduces to Schwarz model (5.36) when \( \delta = 0 \).

The case developed has been purposely chosen to be elementary. Needless to say, in case of large networks and/or including complex components, such as power electronics devices, the possibility to proceed analytically soon becomes impractical and error prone. On the other hand, the implementation of the methods shortly introduced in §5.2.2 into suitable automatic procedures enables a tool such as Matlab-Simulink to easily manage even very complex cases. The visual nature of the program interface allows both a comprehensive control from the user onto the model and a quick implementation and analysis of the modifications that should become necessary.
5.3 Classical Arc Models

Classical black box models for electric arcs are based on the energy conservation principle, written for the arc as an electro-thermal system with an internal source term due to Joule heating and exchanging heat with the surrounding, colder environment. If Joule heating is higher (lower) than energy dissipation to the outer environment, then the arc energetic content increases (decreases). Since the arc electric conductivity is related to the arc energetic content, the electric characterization of the arc as a component of a network is thus produced.

The power balance may be formally translated into the simple ODE

$$\frac{dQ}{dt} = P_{in} - P_{out}, \quad (5.24)$$

where $Q$ is the energy stored in the arc as a thermal system, and is thus measured in $J$. The time derivative of $Q$ is the storage term, while the r.h.s. of $(5.24)$ represents the net power income, i.e., the difference of the incoming thermal power $P_{in}$ minus the outgoing thermal power $P_{out}$ (also termed cooling power), both measured in $W$. Due to hypothesis of Ohmic behavior $(5.2)$, the power inserted into the arc is entirely due to thermal dissipations, or Joule heating, and reads

$$P_{in} = ui = gu^2. \quad (5.25)$$

From this point on, similarities in between black box models end, and specific formulations for the cooling power $P_{out}$ are developed model wise, as described in the next sections. As regards classical models, even though the record of possible cases is wider (see §5.5), we will anyway pay more attention to those models whose structure may be summarized by the first order ODE

$$\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau(g)} \left( \frac{gu^2}{P_{out}(g)} - 1 \right), \quad (5.26)$$

where $\tau, P_{out} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are suitable functions of conductivity.

5.3.1 Mayr Model

One among the most notorious and widely adopted black box model for high-voltage arcs was proposed by Otto Mayr [88, 89], in 1943. Mayr model rests on the following hypotheses and simplifying assumptions:
1. The electric current is relatively low (say, \( i \lesssim 500A \)) and the model is thus suited for describing arcs in the vicinity of current zero, at least not too close to the current zero, as will be shown in §5.4;

2. The arc has the shape of a cylindrical column, whose cross section remains constant in time, since it is unaffected by the power balance (5.24), and its length is only due to the electrode distance (and thus may be assumed constant in many applications);

3. The arc temperature \( T \) is uniform inside the arc column, it is variable in time, and depends on the stored energy \( Q \), since, due to hypothesis 2, the arc cannot expand or contract and its temperature is the only way for the arc to account for the power balance (5.24);

4. The arc conductance \( g \) is uniform within the arc column and depends on the stored energy \( Q \), as a consequence of hypothesis 3, and the dependence law is

\[
g = k \exp(Q/Q_0),
\]

where \( k \) and \( Q_0 \) are constant parameters;

5. The cooling power \( P_{out} \) is constant in time and is equal to its value in steady state conditions;

After hypothesis 4, and particularly thanks to expression (5.27) and time constancy of \( k \), the storage term of (5.24) may be reformulated in terms of arc conductance as

\[
\frac{dQ}{dt} = \frac{d}{dt} \left( Q_0 \ln \frac{g}{k} \right) = Q_0 \frac{d\ln g}{dt} = Q_0 \frac{1}{g} \frac{dg}{dt}.
\]

Plugging back into the energy balance (5.24) and setting \( \tau := Q_0/P_{out} \), which is constant because \( Q_0 \) and \( P_{out} \) are constant, the Mayr arc model is finally obtained and reads

\[
\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau} \left( \frac{gu^2}{P_{out}} - 1 \right),
\]

which is a special case of the classical black box arc model ODE (5.26) for both \( \tau(g) \) and \( P_{out}(g) \) constant. An evidently equivalent formulation of Mayr model reads

\[
\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau} \left( \frac{ui}{P_{out}} - 1 \right),
\]

where Joule heating has been modified by means of Ohm’s law and which is the generally known version.

The Mayr model has two parameters, namely:
5.3. CLASSICAL ARC MODELS

1. \( \tau \), termed the arc time constant;
2. \( P_{\text{out}} \), termed the arc cooling power.

In the spirit of black box modeling, the values of parameters have to be identified by a suitable fitting procedure (see §5.7). This is helpful because it avoids the difficult experimental reproduction of realistic arc condition to directly measure the parameters. On the other hand, the physical meaning of the parameters is not taken into account, which is a drawback because the physics is to some extent lost.

The physical meaning of the \( \tau \) parameter is readily obtained in absence of input power, that is, immediately after the current zero (neglecting the small Joule heating produced by post arc currents), when the linear ODE with constant coefficients

\[
\frac{dg}{dt} = -\frac{g}{\tau}
\]

rules the “free” cooling down of the arc, admitting the solution

\[
g(t) = g_0 e^{-t/\tau},
\]

where \( g_0 \) is the arc conductivity at current zero. In other words, \( \tau \) is a measure for the circuit breaker attitude to recover an electrically insulating state after the current zero. This explains the “time constant” name given to \( \tau \), after the well-known practice in Physics and Engineering to designate with such a term a measure of the time required by an exponentially decaying phenomenon to take place. We recall that one time constant is the time required for the arc to reduce its conductance by approximately 63%, as one may verify by evaluating \( 1/e \), and seven time constants (corresponding to a reduction down to less than \( 1/1000 \) of the original value) is the time conventionally assumed to consider exhausted an exponentially decaying phenomenon.

In the original formulation of Mayr model, the hypothesis is assumed (additionally to those listed above) that the cooling power is entirely due to conduction. Nonetheless, the deduction we have proposed for the model does not make use of such an hypothesis at all. We conclude that this last hypothesis is redundant and thus it is not of mathematical interest.

5.3.2 Cassie Model

In 1939, some years before Mayr, A.M. Cassie proposed a different and to some extent complementary model for arcs in the high-voltage regime [21]. Cassie model rests on the following hypotheses and simplifying assumptions:
1. The electric current is relatively high (say, $i \gtrsim 500A$) and the model is thus unsuited for describing arcs in the vicinity of current zero;

2. The arc voltage is constant in time in the high current period considered, which is by approximation true, apart from fluctuations around a more or less invariant mean voltage value;

3. The arc temperature is constant, both in space (i.e., within the arc zone) and in time;

4. The point wise physical quantities describing the arc, namely electrical conductivity $\sigma$, specific heat, stored energy per unit volume $q$, etc. are constant in time, which is consistent with arc temperature time constancy (hypothesis 3), and provided that pressure dependence of such quantities is neglected;

5. The arc has the shape of a cylindrical column, whose cross section $A(t)$ is variable in time, depending on the arc energy content, so to account for the power balance (5.24) and hypothesis 3, and its length is only determined by the electrode distance $L$ (and thus may be assumed constant in many applications);

6. The specific (per unit volume) cooling power $p_{out}$, measured in $W/m^3$ is constant in time.

The variable arc column cross section in a regime of constant arc voltage allows arc resistance to be adjusted, thus setting the value for the arc current.

In order to obtain the Cassie model, we need to express the energy storage term as a function of arc conductance, similarly to what is done in the case of the Mayr model. A specific (per unit volume) energy content $q$, measured in $J/m^3$, is defined in the arc zone, so that $Q(t) = qA(t)L$. From Ohm’s second law, and accounting for the cylindrical arc shape, one gets $g(t) = \sigma A(t)/L$, whence $A(t)$ can be eliminated. For the sake of clarity, we have marked time dependence in the previous relations, so that time constancy holds for symbols without explicit reference to time $t$, namely $q$ and $\sigma$ (hypothesis 4) and $L$ (hypothesis 3). Summing up, one gets

$$\frac{dQ}{dt} = \frac{d}{dt}(qA(t)L) = \frac{d}{dt} \left( q \frac{g(t)L^2}{\sigma} \right) = \frac{qL^2}{\sigma} \frac{dg}{dt} = Q \frac{1}{g} \frac{dg}{dt},$$

where the relation $q/\sigma = Q/(gL^2)$, following after the definition of $q$ and Ohm’s second law, has been used in the last step.
Plugging back into the energy balance (5.24) and setting $\tau := Q/P_{out}$, an expression formally similar to Mayr’s one is obtained, that reads

$$
\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau} \left( \frac{gu^2}{P_{out}} - 1 \right).
$$

(5.30)

Since $\tau = Q/P_{out} = q/p_{out}$ and both $q$ and $p_{out}$ are constant in time, such is $\tau$, even though $Q$ and $P_{out}$ generally are not. Moreover, since $p_{out}$ is constant, it can be computed in the special case when the cooling power equals Joule heating, i.e., in a reference, “temporally stable” condition. Marking with subscript “0” the special values of variable quantities in this particular condition (which is not necessarily obtained for $t = 0$), one gets $p_{out} A_0 L = P_{out,0} = g_0 u_0^2$, from which

$$
p_{out} = \frac{g_0 u_0^2}{A_0 L}
$$

(5.31)

is obtained.

Expression (5.30) describes an arc model which is only formally similar to Mayr’s. As a matter of fact, arc conductance $g$ is hidden in the total cooling power $P_{out}$, which is not constant (in the general case). Thanks to (5.31), it is possible to reveal such a dependence. First one notices, thanks to Ohm’s second law for a cylindrical conductor, that the ratio $g(t)/A(t)$ equals the constant ratio $\sigma/L$. Therefore also $g(t)/A(t)$ is constant in time and may be evaluated with reference to the temporally stable condition, i.e., $g(t)/A(t) = g_0/A_0$. Accounting for this last relation and remembering (5.31), one gets

$$
P_{out}(t) = p_{out} A(t) L = g u_0^2.
$$

(5.32)

Plugging back (5.32) into (5.30), Cassie arc model is finally obtained and reads

$$
\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau} \left( \frac{u^2}{u_0^2} - 1 \right)
$$

(5.33)

which is a special case of the classical black box arc model ODE (5.26) for $\tau(g)$ constant and $P_{out}(g) = g u_0^2$. Cassie model has two parameters, namely:

1. $\tau$, termed the arc time constant;
2. $u_0$, termed the arc reference voltage drop, in a temporally stable condition.

In the spirit of black box modeling, the values of parameters have to be identified by a suitable fitting procedure (see §5.7). This is extremely helpful for it prevents from experimentally producing a temporally stable condition and measure the relevant arc voltage. On the other hand, the physical meaning of parameters is not exploited for their determination.
In the original formulation of the Cassie model, the hypothesis is (additionally to those listed above) that the cooling power be entirely due to convection. Nonetheless, the deduction we have proposed for the model does not make use of such an hypothesis at all. We conclude that this last hypothesis is redundant and thus it is not of mathematical interest.

5.3.3 Schwarz Model

Mayr and Cassie pioneer works opened a new line of research in arc modeling but still lacked of the required precision to agree with experimental oscillograms [112]. Consequently, many modifications and improvements have been proposed in the literature of black box arc models. Starting from the conceptual framework Mayr arc model (5.28), in 1971 J. Schwarz [135] proposed to introduce a conductance dependence in the expression of $\tau$ and $P_{\text{out}}$. Many authors, viz. Welly [165], Avdonin et al. [5], Hrabovsky and Havel [62], Thiel [149] and Nakamichi and Yuma [106] adopted

$$\tau = \tau_0 g^\alpha$$  \hspace{1cm} (5.34)

and

$$P_{\text{out}} = P_0 g^\beta,$$  \hspace{1cm} (5.35)

due to the capability to adapt to experimental data. Power laws have since been frequently incorporated in the Schwarz arc model

$$\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau_0 g^\alpha} \left( \frac{g u^2}{P_0 g^\beta} - 1 \right),$$  \hspace{1cm} (5.36)

which is clearly a special case of the classical black box arc model ODE (5.26) for $\tau(g)$ and $P_{\text{out}}(g)$ defined by (5.34) and (5.35), respectively. The Schwarz model has four parameters, namely:

1. $\tau_0$, the constant factor in the time constant expression;
2. $\alpha$, the exponent of conductance in the time constant expression;
3. $P_0$, the constant factor in the cooling power expression;
4. $\beta$, the exponent of conductance in the cooling power expression.

In practice, $\tau$ is usually still termed the “time constant” of the arc, despite being no longer a constant. To our knowledge, there is no physical justification neither for expression (5.34) nor for (5.35) and the general consensus in the literature [112] is to consider such functional dependences as mere mathematical
expedients to increase the capability of the relevant arc model to fit experimental oscillograms. As a natural consequence, the four model parameters have to be identified by means of a suitable fitting procedure (see §5.7).

Mayr and Cassie arc models are obtained from Schwarz’s as special cases. Precisely, Mayr model is obviously produced back by removing the functional dependence on conductance from both the time constant and the cooling power, that means, by setting $\alpha = \beta = 0$ and identifying $\tau$ and $P_{\text{out}}$ in (5.28) with $\tau_0$ and $P_0$ in (5.36), respectively. On the other hand, if one sets $\alpha = 0$ and $\beta = 1$ and identifies $\tau$ and $u_0^2$ in (5.33) with $\tau_0$ and $P_0$ in (5.36), respectively, then Cassie model is produced.

Although usually left in oblivion, this last consideration is fundamental because it implies that the mathematical structure of Schwarz ODE (5.36) is an enrichment of both Mayr ODE (5.28) and Cassie ODE (5.33), despite the different physical origin of the latter two models. As a consequence, from the mathematical standpoint Schwarz model adds the fitting capabilities of the Mayr model to those of the Cassie model. If a given oscillogram may be fitted to some degree of precision by Mayr (respectively, Cassie) arc model, then it also may be fitted by the Schwarz arc model to at least the same degree of precision or higher. The reverse implications do not hold in the general case.

Probably the weakest point of the Schwarz model are indeed the choices (5.34) and (5.35). Since there is no physical base leading to power laws, it is highly likely that a good matching with experimental oscillograms is only due to the mathematical properties of such functions, with no guarantee about the possibility of extending fitted parameters to other conditions than those used for their identification. Moreover, the possibility to choose exponents $\alpha$ and $\beta$ in $\mathbb{R}$, and particularly in $\mathbb{R} \setminus \mathbb{N}$, makes the units of $\tau_0$ and $P_0$, i.e., $s \cdot S^{-\alpha}$ and $W \cdot S^{-\beta}$ respectively, be dependent on the values of other parameters.

## 5.4 Non Equilibrium Arc Models

Classical models, such as Mayr, Cassie and Schwarz arc models, are exclusively based on a global thermal balance and the hypothesis of Local Thermal Equilibrium (LTE) is implicitly assumed. The effect of the electrical field can be accounted for by means of non equilibrium theory. When dealing with black box arc modeling, non equilibrium effects may be summarized into a new additive term, proportional to the time derivative of arc voltage. We will thus explicitly term non equilibrium models those including such a corrective term, which has far reaching physical consequences. The underlying physical framework will be the base of the model we propose for low voltage electric arcs in
§5.6.

In developing the application of non equilibrium theory we refer to the black box model proposed in 1966 by W. Rieder and J. Urbanek [125] (and frequently associated to the name of the latter). Even though Rieder and Urbanek developed their model with reference to the high-voltage regime, where strong electric fields are naturally encountered, producing strong deviations from the thermal equilibrium, also in the low voltage realm abandoning the LTE hypothesis still seems to be particularly appropriate in a neighborhood of the current zero, for the plasma contains relatively few charge carriers, so that electron-ion thermalization appears to be prevented. Urbanek model is an improvement of Mayr type models and starts from the set of assumptions listed §5.3.1. Particularly, the arc is supposed to have cylindrical shape with section $A$ and length $L$. In the deduction of the model we will refer to non equilibrium theory, exposed in §3.6.3.

In the arc column, an electric current density of intensity $j$ is produced both by electronic and ionic currents, that is, $j = n_e e^-(v_e + Z_h v_h)$. The latter contribution is usually neglected, since $v_h \ll v_e$. Introducing the electron mobility $\mu_e = v_e / E$ to relate the free electron drift velocity to the driving electric field, we find $j \approx n_e e^- \mu_e E$. By comparison with Ohm’s (first) law $j = \sigma E$ and accounting for Ohm’s second law $g = \sigma A / L$, one gets

$$g = n_e(T_h, T_e) e^- \mu_e A L = g(T_h, T_e). \quad (5.37)$$

Electron temperature $T_e$ is a function of the electric field intensity $E$. Since the electric field vector $\mathbf{E}$ is the gradient of voltage, i.e., $\mathbf{E} = -\nabla u$, its intensity can be approximated as $E \approx u / L$, so that, ultimately, $T_e = T_e(u)$. Therefore $g(t) = g(T_h(t), T_e(u(t)))$, and the total derivative theorem applied to (5.37) yields

$$\frac{1}{g} \frac{dg}{dt} = \frac{d\ln g}{dt} = \frac{\partial \ln g}{\partial T_h} \frac{dT_h}{dt} + \frac{\partial \ln g}{\partial T_e} \frac{dT_e}{du} \frac{du}{dt}. \quad (5.38)$$

The first addendum in the last term is exactly the same that could be deduced in the LTE hypothesis, $T_h$ assuming the role of $T$ as the reference heavy particle gas temperature. The second addendum is the original contribution obtained through non equilibrium theory, accounting for the free electron gas temperature $T_e$ induced by voltage $u$.

The additional term due to non equilibrium effects is termed field correction. The first two derivatives appearing in the field correction can be evaluated analytically, by accurately accounting for the underlying plasma physics, as shown by Rieder and Urbanek [125]. Anyway, in the spirit of black box modeling and in conformity with the treatment reserved to the classical Mayr/Schwarz
portion, it is preferable to encapsulate these two derivatives in an additional parameter $\gamma$. The Urbanek arc model is finally obtained, which reads

$$\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau(g)} \left( \frac{gu^2}{P(g)} - 1 \right) + \gamma \frac{du}{dt},$$

(5.39)

where the fundamental feature, enabling the generalization of classical black box models to non equilibrium theory, is the additional term proportional to the time derivative of the arc voltage.

When dealing with LV arcs in §5.6 we will refer to the two models described in the following two sub-sections. The two models are both special cases of the Urbanek type arc model (5.39), i.e., Mayr type models with the field correction to account for electric field phenomena.

### 5.4.1 Mayr-Urbanek Model

The Mayr model with field correction, or, shortly, the Mayr-Urbanek arc model, reads

$$\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau} \left( \frac{gu^2}{P_{out}} - 1 \right) + \gamma \frac{du}{dt},$$

(5.40)

Mayr-Urbanek model has three parameters, namely:

1. $\tau$, termed the arc time constant;
2. $P_{out}$, termed the arc cooling power;
3. $\gamma$, termed the constant factor in field correction.

Model (5.40) is actually the model proposed by Rieder and Urbanek [125] and is also known as Rieder-Urbanek model or Urbanek model.

### 5.4.2 Schwarz-Urbanek Model

The Schwarz model with field correction, or, shortly, the Schwarz-Urbanek arc model, reads

$$\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau_0 g^\alpha} \left( \frac{gu^2}{P_0 g^\beta} - 1 \right) + \gamma \frac{du}{dt},$$

(5.41)

Schwarz-Urbanek model has five parameters, namely:

1. $\tau_0$, the constant factor in the time constant expression;
2. $\alpha$, the exponent of conductance in the time constant expression;
3. $P_0$, the constant factor in the cooling power expression;
4. $\beta$, the exponent of conductance in the cooling power expression;
5. $\gamma$, termed the constant factor in field correction.

5.5 A Review of Black Box Arc Models

The literature on black box arc models is rich of contributions, many of them developed for HV arcs. Even though LV arcs exist in different physical conditions, it is important to know what modeling choices have been done and which possibilities are offered by black box arc models. Working Group 13.01, Study Committee 13 of CIGRE, compiled a list of commonly used black box models [112], which we hereafter critically review with reference to LV arcs.

Generally speaking, a fundamental virtue of black box models (not only for electric arcs) is to have not too many parameters. As a matter of fact, a high number of degrees of freedom usually allows better fitting capabilities. Nonetheless, if the physical base of the model is poor, it is possible that a many parameter black box model fits the experimental behavior only for a mathematical reason, but there is no theoretical justification to expect that such parameters are still good in different testing conditions than those used for the calibration of the model. In such a very negative condition, model identification would only be a mathematical exercise deprived of any real usefulness.

Before examining the models proposed in the literature, we notice that, thanks to Ohm’s law (5.2), the l.h.s. of virtually all of them allows being reformulated in the “symmetric” form

$$\frac{1}{g} \frac{dg}{dt} = \frac{1}{i} \frac{di}{dt} - \frac{1}{u} \frac{du}{dt}, \quad (5.42)$$

which sometimes proves useful when handling simple arc-network interactions in closed form, as in §2.5.2.

5.5.1 Single ODE Models

A first category of black box arc models includes those expressed by means of a single ODE, such as the classical ones by Mayr (§5.3.1) and Cassie (§5.3.2), their common generalization by Schwarz (§5.3.3) and the non equilibrium theory based corrections (§5.4).
In the framework of Schwarz arc model, Widl et al. [167] proposed to add a constant term to the power laws for the time constant and the cooling power, that is,
\[
\tau(g) = \tau_1 + \tau_0 g^\alpha, \quad P_{out}(g) = P_1 + P_0 g^3,
\]
so that both functions do not vanish when conductance does. The arc model stemming from this assumptions is clearly a generalization of the classical Schwarz model, for \(\tau_1 = P_1 = 0\), and consequently cannot have but better fitting capabilities. Nonetheless, this increases the number of parameters, a fact which is usually aversed, especially in its extreme limit when the operations tends to be arbitrary and difficult to be physically justified.

In 1972, Urbanek [157] proposed to include dielectric breakdown by means of a single ODE which reads
\[
\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau_0} \left( \frac{u_i - P_0}{u_0^2 g} - 1 \right) \left( 1 - \left( \frac{u}{u_d} \right)^2 - \frac{\tau_0}{\tau_0} \left( \frac{du^2}{dt} \right) \right),
\]
where the four parameters \(\tau_0\), \(P_0\), \(u_0\) and \(u_d\) have to be fitted or physically determined. Particularly, \(u_d\) is the dielectric breakdown voltage. Model (5.44) is truly a generalization of the classical model (5.26) with
\[
\tau(g) = \tau_0 \frac{u_0^2 g}{P_0 + u_0^2 g}, \quad P_{out}(g) = P_0 + u_0^2 g
\]
for, when a purely thermal failure is investigated, and so \(u \ll u_d\), the last corrective factor on the r.h.s. of the ODE may be approximated by 1. In the case of LV circuit breakers, since voltage is - indeed - low, the event of a dielectric breakdown is less frequent than thermal restrikes (to which most of our interest is thus devoted).

In 1978, Schwalb et al. [134] proposed a model whose ODE reads
\[
\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau_0 + \tau_1} \left( \frac{u_i}{1 + (g/g_0)^2} - 1 \right) \left( \frac{u}{P_0 + u_0 |i|} - 1 \right),
\]
where the five parameters \(\tau_0\), \(\tau_1\), \(P_0\), \(u_0\) and \(g_0\) have to be fitted or physically determined. Since, by Ohm’s first law, \(i = gu\), model (5.45) is a special case of the classical model (5.26) with
\[
\tau(g) = \frac{1}{\tau_0 + \tau_1} \left( 1 + (g/g_0)^2 \right)^{-1}, \quad P_{out}(g) = P_0 + u_0 |u| g.
\]
In 1980, Portela [118] proposed model for relatively small currents, whose ODE reads

\[
\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau_0} \left( \frac{ui}{a + b|i| + c\sqrt{|i|}} - 1 \right),
\] (5.46)

where the four parameters \(\tau_0, a, b\) and \(c\) have to be fitted or physically determined. Also in the case Portela arc model, thanks to Ohm’s first law, ODE (5.46) may be straightforwardly seen as a special case of the classical ODE (5.26).

A common feature of the models examined so far is the possibility to express the time constant \(\tau\) and the cooling power \(P_{out}\) as specified functions of conductance \(g\) or current \(i\), or both. We have already noticed that Ohm’s first law allows converting conductance dependence into current dependence, or vice versa. All models explicitly referring to conductance dependence are more or less complicated special cases of (5.26). In case current dependence is retained, then the prototype ODE

\[
\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau(|i|)} \left( \frac{g_s(|i|)}{g} - 1 \right)
\] (5.47)

may be considered, where \(g_s(|i|)\) has the meaning of a current depending static conductance, that is, taking values in correspondence of prescribed and constant in time currents \(i\). ODE (5.47) was proposed in Grütz and Hochrainer [51] and by Schmidt [133]. By setting \(P_{out} = i^2/g_s\) and accounting for Ohm’s first law, ODE (5.47) is translated into the form (5.26).

A class of hybrid models, with governing ODE in the form

\[
\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau(g)} \left( \frac{g_s(|i|)}{g} - 1 \right)
\] (5.48)

was proposed in 1986 by Matsumura et al. [86]. Here, conductance and current dependence are both retained and thus model (5.48) is standing in between (5.47) and (5.26).

Finally, von Bonin and Kriechbaum [163] proposed a class of models governed by an ODE of the type

\[
\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau(g)} \left( \frac{ui}{P_{out}(g) - \gamma(g,t)} - \gamma(g,t) \right),
\] (5.49)

with the additional function \(\gamma\) to be found in the high current regime upon the current zero. Model (5.49) can be theoretically seen as a generalization of (5.26), with the additional time dependence, when a cooling power and a time constant formally expressed \(P_{out}(g)\gamma(g,t)\) and \(\tau(g)\gamma(g,t)\), respectively, are plugged into (5.26).
5.5.2 Multiple ODE Models

Models have been proposed that compute arc conductance by solving more than one ODE, each one with its own set of parameters. A first class of such models include those where arc is thought of as a series connections of arc sections [162, 119]. Each section is described by its own conductance, computed from a suitable ODE, and the overall arc conductance is computed as an equivalent series conductance. The idea originates in high-voltage circuit breakers from the axial inhomogeneity of the arc column.

Belonging to this class, we mention the so-called Kema model by van der Sluis et al. [160], accounting to three arc sections ruled by an ODE of type

$$\frac{1}{g_k} \frac{dg_k}{dt} = \frac{1}{\tau_k} \left( \frac{u_k i}{P_k g_k^{2-\lambda_k}} - 1 \right), \quad k \in \{1, 2, 3\}, \quad (5.50)$$

where $g_k$ and $u_k$ are the conductance and voltage, respectively, of the $k$-th arc section, and $\tau_k$, $P_k$ and $\lambda_k$, for $k \in \{1, 2, 3\}$, constitute a set of nine model parameters of which only three are actually free, due to six algebraic constraints imposed on experimental basis. Do to the series connection, the total arc voltage is obtained as the sum

$$u = \sum_{k=1}^{3} u_k$$

over the three arc sections, each one assumed to be ohmic. The same current $i$ flows through all of the arc sections, so that $i = g_k u_k$. The total arc conductance is evaluated as the reciprocal of the sum of the resistances of the arc sections, that is,

$$g = \left( \sum_{k=1}^{3} \frac{1}{g_k} \right)^{-1}.$$

Each section is governed by a classical model. For $\lambda_k = 2$ Mayr model is obtained as a special case, whilst for $\lambda_k = 1$ a model mathematically equivalent to Cassie’s is produced. Three of the above mentioned algebraic constraints on model parameter fix the values of the $\lambda_k$. Particularly, two of them are set equal to 1.9 and 2, which practically amounts to choosing two instances of a Mayr model, which is claimed to be suited for the low current regime. The third $\lambda_k$ is set to 1.4, yielding a model similar to Cassie’s, which is claimed to be suited for the high current regime.

Another idea is that of using different ODE for different periods, typically distinguishing between high and low current regime. We will make use of such a
possibility in our proposal for an arc model for low voltage circuit breakers §5.6. An inspiring example in high-voltage circuit breakers is the model developed by P.H. Schavemaker et al. [130, 132], where the ODE

$$\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau} \left( \frac{ui}{u_0|i|} - 1 \right),$$

(5.51)

with the two parameters $\tau$ and $u_0$, is solved in the high current regime, while the ODE

$$\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau} \left( \frac{ui}{P_0 + P_1ui} - 1 \right),$$

(5.52)

with the three parameters $\tau$, $P_0$ and $P_1$, is solved in the high current regime. The transition in between the two regimes is obtained by merging the two ODE into

$$\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau} \left( \frac{ui}{\max(u_0|i|, P_0 + P_1ui)} - 1 \right),$$

(5.53)

so that, formally, Schavemaker model could also be set into the single ODE model. Equation (5.51) brings some resemblance with Cassie model (5.33), the most noticeable difference being the presence of the square of voltage in this latter model, and it is thus claimed to be suited for describing high currents. Equation (5.52) is an improvement of Mayr model (5.28), with a cooling power linearly dependent on Joule heating, and it is thus claimed to be suited for describing low currents.

5.5.3 Stochastic Models

From the very nature of black box models it is clear that a large number of physical phenomena and physical features of the circuit breaker are not and cannot be included into a simple and integral energy balance. As a result, a purely deterministic model, like those described so far, could be found inadequate to describe the strongly random behavior of the electric arc.

Many ways can be envisaged to introduce stochastic properties. Sporckmann [143] suggested to enrich Mayr model (5.28) with two random functions of time with null average, namely $\zeta_\tau$ and $\zeta_P$, and proposed the ODE

$$\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau_0(1 + \zeta_\tau(t))} \left( \frac{ui}{P_0(1 + \zeta_P(t))} - 1 \right).$$

(5.54)

Legros and Genon [72] proposed the ODE

$$\frac{1}{g} \frac{dg}{dt} = \frac{1}{\tau_0} \left( \left| \frac{u|i|}{k} \right|^{\frac{2}{1+n}} - \chi(t) \right),$$

(5.55)

with the three parameters $\tau_0$, $\alpha$ and $k$ and the random time function $\chi$. 
5.6 A Black Box Model for Low Voltage Arcs

In the low voltage case we propose to model an extinguishing arc by means of an ODE of the classical type \((5.26)\) in the high current regime, switching to an ODE of type \((5.39)\) in the low current regime, when we assume that a cold plasma be present with its peculiar non equilibrium physics. This way of proceeding is definitely a simplification, for the transition from LTE to non equilibrium is reasonably smooth, so that a sharp distinction between the two regimes is probably not rigorously possible. Model identification is here always carried out according to the method described in §5.7. The concepts outlined in this section have been accepted for publication in a journal paper [8].

5.6.1 High Current Regime

We start considering the last 50\(\mu\)s before the current zero, where the arc is initially in a high current regime and seems to progressively extinguish without too many instabilities. In fact, due to the presence of the splitter plates, over a wider time span the plasma column shows a random behavior that prevents us from using black box techniques. Only in the last few microseconds before current zero the LTE hypothesis will be reasonably invalidated.
In this pre-zero period, we made a comparison between Mayr’s model (5.28) and Schwarz’s model (5.36) which have been widely used by several authors [144, 112]. When identifying model parameters, the pre-zero period is considered only. Experimental data are compared with results obtained from Mayr and Schwarz models in Figure 5.2, for the pre-zero region only. Mayr’s model is fairly good but using Schwarz’s equation the final result is significantly improved. Anyway, if it were only for the pre-zero period, both models could be used.

5.6.2 Low Current Regime

To study the low current regime we start considering a time span of 50µs after the current zero, since it is apparently long enough to appreciate either an extinguishing or a diverging post-arc current, depending on the ability of the breaker to interrupt the current. In case of successful interruption, the current is limited to a few ampere and in case of failures the low current regime is attained during most of the time span considered. In this period we focused on the effects of the field correction on Mayr’s and Schwarz’s model (5.40) and (5.41), respectively.

We consider successful interruptions first. Figure 5.3 shows the effects of the field correction applied to Mayr’s model for the post-zero period, while Figure 5.4 shows the same effects applied to Schwarz’s equation. Experimental data are reported in both figures. When identifying model parameters, the post-zero period is considered only. From these comparisons we see that, with the field correction, the simulated arc current is closer to the experimental one. The agreement between the predictions of non equilibrium models and experimental data is striking. Mayr’s model is acceptably good in the high current regime. When it is enhanced by the field correction in the low current regime, it is able to reproduce the experimental data quite well as long as successful interruptions are considered.

The picture is very different when re-ignitions are considered. Classical models without field correction are seen to be inadequate. In Figure 5.5, experimental data are compared with the predictions of both Mayr and Schwarz models with field corrections. Mayr-Urbanek equation (5.40) fails to describe the post-zero phase. Schwarz-Urbanek equation (5.41), instead, is still able to fit the measured data and is therefore to be preferred. We have tested many of the models described in §5.5 and the field correction only proved so successful in fitting experimental data. Thanks to the underlying theoretical reason that justifies such a correction, there is rather strong evidence that the proposed one be an adequate model for low voltage arcs.
Figure 5.3: Post-zero arc current with interruption: experimental (star), computed with Mayr model (dashed) and Mayr model with field correction (solid).

<table>
<thead>
<tr>
<th>Model</th>
<th>$\tau_0$</th>
<th>$\alpha$</th>
<th>$P_0$</th>
<th>$\beta$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mayr</td>
<td>7.60e-5</td>
<td>-</td>
<td>3.02e+3</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Mayr + $\frac{du}{dt}$</td>
<td>4.99e-5</td>
<td>-</td>
<td>2.15e+3</td>
<td>-</td>
<td>-4.10e-6</td>
</tr>
</tbody>
</table>

Figure 5.4: Post-zero arc current with interruption: experimental (star), computed with Schwarz model (dashed) and Schwarz model with field correction (solid).

<table>
<thead>
<tr>
<th>Model</th>
<th>$\tau_0$</th>
<th>$\alpha$</th>
<th>$P_0$</th>
<th>$\beta$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schwarz</td>
<td>8.32e-5</td>
<td>0.51</td>
<td>3.19e+3</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>Schwarz + $\frac{du}{dt}$</td>
<td>5.03e-5</td>
<td>0.03</td>
<td>2.32e+4</td>
<td>0.53</td>
<td>-4.78e-6</td>
</tr>
</tbody>
</table>
5.6.3 High and low current regime merging

In the previous sections we identified two sets of black box parameters for the pre and post zero period and compared the simulated and experimental arc current for the two phases, separately. The study was introductory for modeling high and low current regimes. Since the regime transition takes place across the current zero, based on the intensity of current, the merger of the two phases is done by performing a simulation over the entire pre and post periods, switching from one differential equation to the other at a certain time. Assuming the non equilibrium effects to be dominant for a small enough current, we propose to use a current threshold to define the switching time at which the field correction should be introduced. By construction, the transition from one model to the other occurs (shortly) earlier than precisely at current zero. Precious information worthy for further investigations may be hidden in this behavior.

The current threshold is found through a standard least square optimization between the experimental curve and the one obtained integrating the two models. Typical values for the threshold range from 0.5 A to 1.5 A. Figure 5.6 and 5.7 show the results of the merging procedure in case of successful interruption and thermal re-ignition, respectively. Both are done using Schwarz’s model.
in its original form for the pre-zero phase, equation (5.36), and its enhanced version by field correction for the post-zero region, equation (5.41). For a given current zero, we denote by $t_t$ the time when current magnitude first descends below the above threshold. For the sake of simplicity, the high current regime is never restored after switching to low current, despite the possible overcoming of the current threshold.
In conclusion, the ODE we propose to analyze low voltage arcs reads

\[
\frac{1}{g} \frac{dg}{dt} = \begin{cases} 
\frac{1}{\tau_{0h} g^{\alpha_h}} \left( \frac{g u^2}{P_{0h} g^{\beta_h}} - 1 \right) & \text{for } t < t_1 \\
\frac{1}{\tau_{0l} g^{\alpha_l}} \left( \frac{g u^2}{P_{0l} g^{\beta_l}} - 1 \right) + \gamma \frac{du}{dt} & \text{otherwise}
\end{cases}
\]

for \(t < t_1\).

The proposed model for low voltage arcs has four parameters for the high current (or equilibrium) regime, namely:

1. \(\tau_{0h}\), the constant factor in the time constant expression;
2. \(\alpha_h\), the exponent of conductance in the time constant expression;
3. \(P_{0h}\), the constant factor in the cooling power expression;
4. \(\beta_h\), the exponent of conductance in the cooling power expression;

and five parameters for the low current (or non equilibrium) regime, namely:

1. \(\tau_{0l}\), the constant factor in the time constant expression;
2. \(\alpha_l\), the exponent of conductance in the time constant expression;
3. \(P_{0l}\), the constant factor in the cooling power expression;
4. \(\beta_l\), the exponent of conductance in the cooling power expression;
5. \(\gamma\), termed the constant factor in field correction.

The four (five) parameters for the high (low) current regime are identified based on the 50μs time interval before (after) the current zero.

After fitting experimental oscillograms with the proposed model, we conclude that, thanks to non equilibrium theory, it is possible to deduce a physically consistent black box model for low voltage arcs. Such a model distinguishes between a pre-zero phase, when a hot plasma is burning in LTE conditions, and a post-zero phase, when a cold plasma close to extinction is characterized by non equilibrium effects. It is possible to set this two-stage model in the framework of black box modeling, by an additive correction to classical Schwarz model. The fitting of experimental oscillograms proves to be extremely satisfactory, with a negligible computational burden. Testing the procedure on real industrial circuit breakers, manufactured by ABB, clearly shows the improvements achieved by accounting for non equilibrium effects with reference to classical black box models. Particularly, in the case of failed
interruptions and current restrikes the role played by non equilibrium effects seems to be fundamental. This conclusion is in agreement with known results for high-voltage arcs [125]. The novelty is the applicability to the low voltage realm.

5.7 Model Identification

After defining a suitable black box model, which reasonably accounts for the major physical issues underlying the arc behavior, the question arises as to estimate the value of its parameters. We hereafter propose a mathematical algorithm for model identification, solving a constrained optimization problem by suitably coupling gradient moves with heuristic search methods. The basic core of the algorithm is the classical Levemberg-Marquardt algorithm [74, 85] for nonlinear least squares in the unconstrained case. Our proposal to handle the constrained case exploits line search and ray tracing [46] to detect and not trespass the boundary of the feasible region. The concepts outlined in this section have been accepted for publication in a journal paper [8].

5.7.1 Problem Setting

The starting point is the second equation of (5.1), where arc current is eliminated thanks to Ohm’s law $i = gu$ and the experimental arc voltage signal is inserted into function $\varphi$, as already described in §5.2. Whenever needed, the simulated arc current is numerically computed by solving the black box arc ODE for the arc conductance $g$, and then by using Ohm’s law again. After arc current elimination and experimental arc voltage insertion into $\varphi$, this latter function becomes a function of time, conductance and model parameters only, which we will indicate as $f$.

Summing up, the general form of black box models (5.26) and (5.39) may be rearranged as

$$\frac{dg}{dt} = f(t, g(t, \theta), \theta),$$

where $\theta \in \Theta \subseteq \mathbb{R}^D$ is a vector collecting the $D$ model parameters. Physical or modeling issues generally prescribe bounds on parameters, so that they are defined over a domain $\Theta$, termed the feasible region. Due to the generality of the form (5.57), the method we propose can be conveniently adopted for parameter identification of black box models under very general conditions.
Very frequently, the feasible region has the form of the tensor product

$$\Theta = \bigotimes_{j=1}^{D} \Theta_j$$

(5.58)

where $\Theta_j$ is either a bounded interval of the type $[\theta^l_j, \theta^u_j]$, or a half bounded interval of either the type $(-\infty, \theta^u_j]$ or the type $[\theta^l_j, +\infty)$. Real, prescribed parameters $\theta^l_j$ and $\theta^u_j$ have the meaning of lower bounds and upper bounds, respectively. Nonetheless, the method hereafter proposed does not necessarily require that $\Theta$ have the special, simple form (5.58). The feasible region may have a very general form. Also, neither convexity nor simple connectedness are strictly required. On the other hand, we assume that $\Theta$ be closed, so that the identified parameters may also lie on the boundary $\partial \Theta$ of the feasible region. The case of an open, or partially open, set $\Theta$ may always be included, with some negligible approximation, by a slight contraction of the feasible region in those zones where $\theta \in \partial \Theta \Rightarrow \theta \notin \Theta$. Finally, we assume (as always implicitly done in the theory of model identification) that $\Theta$ be a complete, or Cauchy, space, i.e., a metric space where any Cauchy sequence of points in $\Theta$ converges to a point in $\Theta$ [147]. Intuitively and fancily, this means that $\Theta$ is not missing any point, neither inside nor at the boundary.

We will denote with $\hat{g}_k$ and $\dot{\hat{g}}_k$ the experimental sample at time $t_k$ of arc conductance and of its time derivative, respectively. On the other hand, we term $\hat{g}_k(\theta) = \hat{g}(t_k, \theta)$ the approximate estimate of conductance at time $t_k$, obtained by means of the black box model with parameter set $\theta$.

The main idea of the fitting procedure is to suitably define a residual $r_k(\theta)$, at time $t_k$, when the conductance evaluated by means of the black box model, with parameters $\theta$, and the experimentally observed one are plugged into the differential equation (5.57). Depending on how the latter concept is implemented, different methods are obtained. Particularly, we have defined a holonomic and a nonholonomic fitting, later described. A good practice is to start with the holonomic method, which is faster albeit less accurate, to roughly locate a neighborhood of the optimum point, and then to refine the search by means of the nonholonomic method.

### 5.7.2 Nonlinear Least Squares (NLLS)

Both in the case of the holonomic and the nonholonomic method, a nonlinear least square approach is adopted to deduce the best choice of model parameters. Therefore, in order to measure the global discrepancy between experimental
data and black box prediction, the goal function

\[ F(\theta) := \frac{1}{2}\|r(\theta)\|^2 \]  

(5.59)
is defined, where \( r(\theta) := \sum_{k=1}^{N}r_k(\theta) \) is a vector collecting the \( N \) residuals time wise, whilst \( \| \cdot \|_2 \) is the \( \ell^2 \) norm. The optimal parameter set \( \hat{\theta} \) is chosen such that

\[ \hat{\theta} = \arg \min_{\theta \in \Theta} F(\theta). \]  

(5.60)

5.7.3 Levenberg-Marquardt Algorithm (LMA)

The parameter identification problem is then cast to the nonlinear, constrained optimization (5.60), which we address with the Levenberg-Marquardt algorithm (LMA) [74, 85]. The suitable coupling of the LMA with heuristic search methods allows to account for the particular shape of the feasible region \( \Theta \).

The LMA is iterative, the basic iteration being a gradient move over a suitably defined \( D \)-manifold. We first compute the derivative of the goal function with reference to the \( j \)-th parameter, that reads

\[ \frac{\partial F}{\partial \theta_j}(\theta) = \sum_{k=1}^{N}r_k(\theta)\frac{\partial r_k}{\partial \theta_j}(\theta), \quad j \in \{1, \ldots, D\}. \]  

(5.61)

This way, the analytical expression is available for the goal function gradient \( \nabla F(\theta) = J(\theta)^T r(\theta) \), where \( J(\theta) = \nabla r(\theta) \) is the Jacobian of the residual vector as a function of \( \theta \). The Jacobian can be explicitly arranged in matrix form as

\[ J_{kj} = \frac{\partial r_k}{\partial x_j}(\theta), \quad k \in \{1, \ldots, N\}, \quad j \in \{1, \ldots, D\}, \]  

(5.62)

where \( N \) is the number of samples for the conductance time history. The availability of analytical derivatives quickens the execution and increases the precision, if compared to finite differencing. Then, differencing a second time, one gets

\[ \frac{\partial^2 F}{\partial \theta_j \partial \theta_h}(\theta) = \sum_{k=1}^{N} \left( \frac{\partial r_k}{\partial \theta_j} \frac{\partial r_k}{\partial \theta_h} + r_k \frac{\partial^2 r_k}{\partial \theta_j \partial \theta_h} \right)(\theta), \]  

(5.63)

so that the exact expression of the Hessian of the goal function reads

\[ H(F)(\theta) = J(\theta)^T J(\theta) + \sum_{k=1}^{N} r_k(\theta)H(r_k)(\theta), \]  

(5.64)
where $H(r_k)(\theta)$ is the Hessian of the $k$-th residual. The Taylor series expansion of the goal function, truncated to the second order and centered in a point $\theta_\ell \in \Theta$ (coming from the previous iteration of the LMA), reads

$$F(\theta) = F(\theta_\ell) + \nabla F(\theta_\ell)^T \Delta \theta_\ell + \frac{1}{2} \Delta \theta_\ell^T H(F)(\theta_\ell) \Delta \theta_\ell + o(\|\Delta \theta_\ell\|^2),$$

(5.65)

where $\Delta \theta_\ell := \theta - \theta_\ell$ and $o(\|\Delta \theta_\ell\|^2)$ is Peano’s form of the remainder of the series. The LMA iteration updates the parameter set estimate $\theta_\ell$ with a new, better estimate $\theta_{\ell+1}$ by minimizing the equation of the $D$-manifold

$$\Phi_\ell(\theta) := F(\theta_\ell) + r(\theta_\ell)^T J(\theta_\ell) \Delta \theta_\ell + \frac{1}{2} \Delta \theta_\ell^T (J(\theta_\ell)^T J(\theta_\ell) + \mu I_D) \Delta \theta_\ell,$$

(5.66)

which is an approximation of the exact expression, (5.65), obtained by replacing the correct expression of the Hessian with the first addendum only on the r.h.s. of (5.64) and by adding the stabilizing term $\mu I_D$, with $\mu \in \mathbb{R}$ and $\geq 0$. In the practice, high values of $\mu$ are seen to be beneficial for the stability of the method, whilst low values of $\mu$ allow faster convergence (if any). When $\mu$ is set to zero, the Gauss-Newton algorithm (GNA) \cite{11} is obtained as a special case. A compromise value of $\mu$ may be chosen on a heuristic base, or it may be modified adaptively, e.g., by starting with a cautious choice, such as the GNA, and then accelerating progressively, according to some suitable criterion.

Minimizing (5.66), which is a quadratic $D$-manifold, leads to the well-known system of normal equations, whose solution is the descent direction

$$p_\ell = -(J_\ell^T J_\ell + \mu I_D)^{-1} J_\ell^T r_\ell.$$

(5.67)

From (5.67), one immediately understands that the LMA is a Tychonoff regularization \cite{154, 155, 156} of the GNA. As a matter of fact, the Moore-Penrose pseudoinverse $J_\ell^\dagger$ \cite{101, 116} of the matrix $J_\ell \in \text{Hom}(\mathbb{R}^D, \mathbb{R}^N)$ is obtained as the limit

$$J_\ell^\dagger = \lim_{\mu \to 0} (J_\ell^T J_\ell + \mu I_D)^{-1} J_\ell^T.$$

The $(\ell + 1)$-th estimate of the optimal parameter set $\hat{\theta}$ is finally obtained as

$$\theta_{\ell+1} = \theta_\ell + \alpha_\ell p_\ell,$$

(5.68)

where the real scalar $\alpha_\ell \geq 0$ has to be determined by solving a monodimensional optimization problem, as later detailed.

The starting point for LMA iterations may be obtained by some heuristic exploration method of the feasible region. A good starting point is such that
the LMA converge to a global minimum (notice that, though rather unlikely, global minima may even be multiple, and obviously equal valued, so that one should talk about “a” and not “the” global minimum; from a perturbative approach standpoint, multiple global minima have to be treated as local minima, anyway). Convergence to a global minimum is not necessarily the case for descent methods when the goal function exhibits multiple local minima. Convergence to a local minimum which is not a global minimum, though unpleasant, needs not be catastrophic, provided the relevant parameter set yields a not too different value of the goal function from the really minimal one. In the end, model identification is a compromise choice, based on a rather arbitrarily chosen criterion. Physical insight on model parameters as well as experience from fitting similar oscillograms may be helpful in spotting a plausible starting point. Otherwise, genetic algorithms may be used as a preliminary method.

5.7.4 Line Search

After the descent direction $p_\ell$ has been found, the question arises as to find a suitable advancement $\alpha_\ell p_\ell$ along such a direction, so to produce a better estimate $\theta_{\ell+1}$ than $\theta_\ell$, according to (5.68). The problem is solved by choosing an appropriate line search method [109].

The value of the parameter $\alpha_\ell$, which is generally different iteration wise, must account for the difference between the real shape of $D$-manifold $F$ compared to its approximating $D$-manifold $\Phi_\ell$. The minimum of the latter could be not acceptable in order to minimize the former. We are concerned with the sections of the $D$-manifolds along the search direction $p_\ell$ returned by the $\ell$-th iteration of the LMA. Such sections are 1-manifolds with boundary (see Figure 5.8), one obvious limit being the point $\theta_\ell$ (relevant to $\alpha_\ell = 0$) from the previous LMA iteration. Moreover, an upper bound on $\alpha_\ell$ could derive from intersecting $\partial \Theta$ (black circle in Figure 5.8). This latter point can be found very straightforwardly, e.g., by a ray tracing routine [46].

Our strategy for finding the new update $\theta_{\ell+1}$ is to start from an initial guess and progressively reduce the value of $\alpha_\ell$ until a suitable admissibility condition is met, producing the sequence of candidate updates (indicated in Figure 5.8 by crosses on the section of $\Phi_\ell$ and, when internal to $\Theta$, by white circles on the section of $F$). First, we require feasibility (e.g., the rightmost cross in in Figure 5.8 is outside $\Theta$ and thus cannot be accepted). By construction, the coupling of the LMA with our line search is an interior point method. Second, we want the new update $\theta_{\ell+1}$ be better than $\theta_\ell$ and so we require the fulfillment of Armijo’s condition (also known as Wolfe’s first condition), which reads [109]

$$F(\theta_\ell + \alpha_\ell p_\ell) \leq F(\theta_\ell) + c\alpha_\ell \nabla F(\theta_\ell)^T p_\ell,$$

(5.69)
Figure 5.8: Line search method: the two 1-manifolds obtained by sectioning the original $D$-manifold $F$ and its approximation $\Phi_\ell$ along the search direction $p_\ell$ are shown with solid lines (bold and thin, respect.), the tangent and the secant lines with dash-dot and dashed lines, respect., and the sequence of candidate updates by white circles and crosses.; the black circle is the upper bound obtained by intersecting the boundary of the feasible region.

where $c \in (0,1)$. Geometrically, condition (5.69) requires that the new update be below the secant line (dashed in Figure 5.8) obtained by reducing (in modulus) by a factor $c$ the (negative) slope of the common tangent line (dash-dot in Figure 5.8) for $\theta_\ell$ to both the 1-manifolds.

The new update $\theta_{\ell+1}$ needs not be optimal, but subsequent LMA iterations will produce better updates, possibly but not necessarily along other descent directions, until convergence (which we heuristically accept whenever the goal function decrement is small relatively to some quantity, suitably deduced by experimental conductance data and according to the holonomic or nonholonomic fitting procedures). During the line search, $\alpha_\ell$ is reduced by multiplication by a heuristic coefficient $\delta_1 \in (0,1)$, which ensures exponential velocity of adaptation. The initial guess for $\alpha_\ell$ is obtained multiplying by a coefficient $\delta_2 > 1$ the last $\alpha_\ell$ value used in the previous outer iteration. If initial guesses produce acceptable updates, this procedure ensures exponential velocity of adaptation.

Formally, if we term $\alpha_{\ell}^{(m)}$ the $m$-th update of the line search parameter $\alpha_\ell$
5.7. MODEL IDENTIFICATION

during the $\ell$-th outer LMA iteration, then

$$\alpha^{(m)}_{\ell} = \delta^m_1 \alpha^{(0)}_{\ell}$$

describes the inner exponential contraction of the line search parameter inside a single step of the LMA, whilst

$$\alpha^{(0)}_{\ell+1} = \delta^2_2 \alpha^{(m)}_{\ell}$$

describes the choice of the new initial guess for inner looping, where $m_\ell$ is the number of inner iterations needed to reach a good update and advance to the next LMA outer iteration. Finally, in case $s_\ell$ consecutive outer LMA iterations do not require reducing the value of the line search parameter after the $\ell$-th LMA iteration has been completed, then

$$\alpha^{(0)}_{\ell+s_\ell} = \delta^{s_\ell}_2 \alpha^{(m_\ell)}_{\ell}$$

describes the exponential expansion of the line search parameter.

5.7.5 Holonomic Fitting

The model identification method outlined so far is very general. Now we show how the general formulation meets black box modeling, by providing a suitable way to define the residual function $r(\theta)$. The choice is not unique.

A simple way to define residuals is to plug experimental samples of conductance and of its time derivative on the r.h.s. and l.h.s., respectively, of (5.57). The $k$-th residual is then the modeling error

$$r_k(\theta) := \dot{\tilde{g}}_k - f(t_k, \tilde{g}_k, \theta), \quad \forall k \in \{1, \ldots, N\}. \tag{5.70}$$

Since time derivatives of experimental conductance are required, a suitable smoothing technique is mandatory. We have chosen Savitsky-Golay filtering [129], as described in §2.3. The method stemming out of this choice is termed holonomic because the goal function may be defined in terms of non differential constraints.

5.7.6 Nonholonomic Fitting

Higher accuracy may be achieved by accounting for the differential relationship linking conductance to its time derivative and thus following the evolutionary nature of black box models. Particularly, we define the $k$-th residual by directly
comparing experimental conductance with its counterpart predicted by the black box model, that is,

\[ r_k(\theta) := \tilde{g}_k - \hat{g}_k(\theta), \quad \forall k \in \{1, \ldots, N\}. \]  

(5.71)

Since experimental data are unaffected by model parameters, the Jacobian (5.62) reads

\[ J_{kj}(\theta) = \frac{\partial}{\partial \theta_j} (\tilde{g}_k - \hat{g}_k(\theta)) = -\frac{\partial \hat{g}_k}{\partial \theta_j}(\theta), \quad \forall k \in \{1, \ldots, N\}, \forall j \in \{1, \ldots, D\}. \]  

(5.72)

We now need to evaluate the Jacobian, time wise. By time derivation of (5.72), then by swapping the order of derivatives by Schwarz’s theorem and finally by recalling (5.57), one gets

\[ \dot{J}_{j} = -\frac{\partial^2 \hat{g}}{\partial t \partial \theta_j} = -\frac{\partial^2 \hat{g}}{\partial \theta_j \partial t} = -\frac{d\hat{f}}{d\theta_j}, \quad \forall j \in \{1, \ldots, D\}, \]  

(5.73)

where we have introduced the shorthand notation \( \hat{f} = f(t, \hat{g}, \theta) \) for the r.h.s. of black box equation (5.57), written in terms of the predicted conductance, and where time dependence has been restored to full generality (i.e., the subscript \( . \) in place of \( k \) in \( J_j \) and \( \hat{g} \), which stands for a continuous dependence from the real time variable, instead of sampling in correspondence of an integer time variable). In the case at hand, the total derivative theorem reads

\[ \frac{d\hat{f}}{d\theta_j} = \frac{\partial \hat{f}}{\partial \theta_j} + \frac{\partial \hat{f}}{\partial \hat{g}} \frac{\partial \hat{g}}{\partial \theta_j}, \quad \forall j \in \{1, \ldots, D\}, \]  

(5.74)

so that, by substituting (5.74) into (5.73) and recalling (5.72), the following set of first order ODE is obtained

\[ \dot{J}_{j}(t) = -\frac{\partial \hat{f}}{\partial \theta_j}(t) + \frac{\partial \hat{f}}{\partial \hat{g}}(t)J_{j}(t), \quad \forall j \in \{1, \ldots, D\}, \]  

(5.75)

establishing a differential characterization of each entry of the jacobian matrix. The problem of computing the Jacobian is well posed if a suitable initial condition is provided (Cauchy initial value problem). We arbitrarily choose an initial time instant and we impose that the black box model prediction of conductance equal the relevant experimental datum. Since this means that any subsequent evolution of conductance, regardless the value of \( \theta \), must start from that given sample point, then the initial value of the residual (5.71) identically vanishes and is therefore independent of \( \theta \). Thus the derivative of the Jacobian with reference to any model parameter is null when evaluated precisely at the initial instant and homogeneous initial conditions are assumed when solving the Cauchy initial value problem.
Due to the nonlinear nature of ODE (5.75), the solution to the relevant initial value problem may be carried out numerically by one of the many methods available, such as Runge-Kutta (possibly the classical Runge-Kutta-Fehlberg RKF45, with adaptive time stepping) or even the simple explicit Euler method. In case finer time stepping then sampling rate is required, linear interpolation in between consecutive samples may be assumed. The procedure so far described is termed nonholonomic, because differential constraints cannot be excluded from the formulation of the method.

5.8 Results

In §5.6 we have described a black box model for low voltage arcs which seems to reproduce the current zero very well. The fitting procedure to obtain the free parameters was also presented, and some results were given. Now we would like to quantify the ability of our model to describe the arc shortly before and after the current zero in quantitative way. To do that, we focus on the performance indicators defined in §2.5, which were originally meant to judge the interruption quality of a test. In fact, comparing the value of an indicator evaluated using experimental signals, simply denoted with “experimental indicator”, with the value that it assumes when evaluated using the simulated signal, or “simulated indicator”, we can have an idea of how the fitted curves are close to the experimental ones. Also, in the next sections, we will address the question of the stability of the black box parameters, i.e., how much they vary when calculated on nominally identical tests.

5.8.1 Black box arc parameters

In §2.5.3 we tested 30 breakers, divided 3 groups, characterized by an increasing test voltage, of 10 specimens. Now we will apply our black box model and the related fitting technique to extract the arc parameters of these 30 experiments. First of all we recall the main result of that statistic campaign, namely that a great spread affects low voltage interruptions even when dealing with nominally identical specimens in the very same testing conditions. We stress that with “nominally identical” we refer to specimens which result to be the same according to the tolerances accepted by industrial manufactures. Then, even among the same group of 10 breakers tested in the same conditions, we can expect the arc parameters to be quite different simply because the breaker itself behaves according a wide-spread statistics. From this point of view, every reasonably simple mathematical model will probably inherit this “original sin”. The results are summarized in Table 5.1, where the nine parameters (four for
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<table>
<thead>
<tr>
<th>Model Parameter</th>
<th>Voltage Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>340V</td>
</tr>
<tr>
<td>(\tau_{0h})</td>
<td>(1.23±0.29) \cdot 10^{-5}</td>
</tr>
<tr>
<td>(\alpha_h)</td>
<td>0.39±0.07</td>
</tr>
<tr>
<td>(P_{0h})</td>
<td>(8.47±2.12) \cdot 10^{4}</td>
</tr>
<tr>
<td>(\beta_h)</td>
<td>0.67±0.11</td>
</tr>
</tbody>
</table>

| Black box model for the post-current zero regime |
| \(\tau_{0l}\)   | (5.02±0.01) \cdot 10^{-5} | (5.01±0.01) \cdot 10^{-5} | (5.02±0.01) \cdot 10^{-5} |
| \(\alpha_l\)    | 0.24±0.16      | 0.44±0.13      | 0.39±0.22      |
| \(P_{0l}\)      | (9.53±3.94) \cdot 10^{4} | (8.18±6.75) \cdot 10^{4} | (5.20±3.99) \cdot 10^{4} |
| \(\beta_l\)     | 0.74±0.34      | 0.71±0.32      | 0.54±0.36      |
| \(\gamma\)      | (-0.90±0.55) \cdot 10^{-3} | (-1.50±0.70) \cdot 10^{-3} | (-1.50±0.50) \cdot 10^{-3} |

Table 5.1: Average and standard deviation of model parameters at low, medium and high testing condition severity.

The pre-zero phase and five for the post-zero phase) are listed, each one with its mean value and standard deviation. As anticipated, the latter is quite big, especially for what concerns the post-zero phase. Anyway, with tests that are ab origine so different even in the same testing conditions, it is hard to say if the mathematical identification of the parameters is affected by a non-uniqueness problem. This, indeed, is a possibility that we cannot exclude a priori and it will be part of our future investigations. Figures from 5.9 to 5.17 show every parameter as a function of the test voltage. In our opinion the results are too unstable to draw any conclusion about a possible correlation between the parameters and the severity of the test. Such a correlation, anyway, is crucial when using a black box approach to predict the outcome of an experiment, whose testing conditions will generally differ from those used to extract the arc parameters.

Further research will be done, also, to understand which values are admissible of each parameter. For instance, for its physical interpretation the cooling power, \(P_0\), cannot be negative, and so is the time constant, \(\tau_0\). Anyway, finding suitable bounds for the remaining parameters is a much more complicate task. In our opinion, such bounds should be found in such a way to allow the black box differential equation to admit both the successful re-ignition and the (thermal) failure. For example, it can be shown that, for certain values of the exponent of the time constant, \(\alpha\), the conductance \(g\) cannot approach the zero value. This would mean that the arc will never extinguish for a purely mathematical reason. Therefore, if bounds are not taken into account, it may happen that a black box equation, fitted on experimental data coming from a failure, could not be able to describe an extinguishing arc when it is inserted into a
Figure 5.9: Parameter $\tau_0h$ at low, medium and high testing condition severity, with average and standard deviation (white markers denote successes, black denote failures.

Figure 5.10: Parameter $\alpha_h$ at low, medium and high testing condition severity, with average and standard deviation (white markers denote successes, black denote failures.)
Figure 5.11: Parameter $P_{bh}$ at low, medium and high testing condition severity, with average and standard deviation (white markers denote successes, black denote failures.

Figure 5.12: Parameter $\beta_{bh}$ at low, medium and high testing condition severity, with average and standard deviation (white markers denote successes, black denote failures.
Figure 5.13: Parameter $\tau_0$ at low, medium and high testing condition severity, with average and standard deviation (white markers denote successes, black denote failures).

Figure 5.14: Parameter $\alpha_1$ at low, medium and high testing condition severity, with average and standard deviation (white markers denote successes, black denote failures.)
Figure 5.15: Parameter $P_0$ at low, medium and high testing condition severity, with average and standard deviation (white markers denote successes, black denote failures).

Figure 5.16: Parameter $\beta_l$ at low, medium and high testing condition severity, with average and standard deviation (white markers denote successes, black denote failures.)
5.8. RESULTS

Figure 5.17: Parameter $\gamma$ at low, medium and high testing condition severity, with average and standard deviation (white markers denote successes, black denote failures.

lumped-parameter network and the severity of the test is reduced. Since this kind of analysis is the core of black box modeling, this issue will deserve further investigations, provided that black box modeling is recognized as suitable tool for low voltage circuit breaker development.

5.8.2 Theoretical Calculation of Performance Evaluators

The most remarkable aspect of our black box model is, in our opinion, the ability to reproduce the behavior of the arc shortly before and after current zero, either in case of successful interruption or failure. The ability of such a model has been briefly discussed in §5.6, but now we want to approach this issue in a more quantitative way. For this reason we choose the two performance indicators $\eta_7$ and $\eta_8$, namely the charge flown through the post-arc channel in the first 10 $\mu$s after the current zero and the corresponding Joule’s integral. We use them not to evaluate the interruption quality but to judge the fitting procedure by simply comparing the experimental value of such indicators with the relevant, simulated value based on the calculated arc current. This comparison is done using the tests described in §2.5.3, which were initially meant to study the repeatability and stability of low voltage interruptions. Table 5.2 summarizes the results, along with Figure 5.18 and 5.19, where squares denote
experimental values, while circles denote values computed with the low voltage arc black box model. As one can readily see, the fitting procedure is satisfactory, especially when the test severity is under the failure threshold, both for what concerns the mean values of the indicators and their standard deviation. Once again, we want to stress that the statistical spread that affects these results is not due to the arc model, but it is inherited by the intrinsic stochastic behavior of low voltage interruption.

The good results obtained with the fitting procedure applied to our black box model is surely encouraging, but several studies are required to proceed in this analysis. One of these is the correct bounding of the arc parameters that characterize the two differential equations for the conductivity, as discussed before. Then, the most challenging achievement is to transform what is now a model to reproduce an interruption into a model to predict the outcome of a test. This will be possible only studying the interaction between the arc model with a suitable model of the electric network used to test low voltage circuit breakers.

We see three major difficulties in such an analysis. First, the low-voltage arc interruption is affected by a remarkable statistical spread, which requires a time-consuming (and relatively expensive) experimental campaign for the determination of the arc parameters. Second, it seems unlikely that simple models such as those investigated in this work, and particularly leaving the breaker mechanism and other specific features out, could be powerful enough to predict the dependence on the testing conditions. Since the role of the latter appear as crucial, either new and possibly complex approaches need to be investigated, or several “local” models, that is, with a scope restricted to specific testing conditions, need to be developed, so to cover the whole range of industrially interesting conditions. Third, the real arc-network interaction around the current zero can be very different from the simulated behavior, as shown in §2.2.2. Since we want to develop an industrially useful approach, so that we are interested in real networks, a very accurate and reliable model is

<table>
<thead>
<tr>
<th>Performance Evaluator</th>
<th>340V</th>
<th>380V</th>
<th>500V</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_7 ) [C]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exp.</td>
<td>(1.45±0.53) \cdot 10^2</td>
<td>(2.72±0.73) \cdot 10^2</td>
<td>(5.61±1.48) \cdot 10^2</td>
</tr>
<tr>
<td>Sim.</td>
<td>(1.51±0.61) \cdot 10^2</td>
<td>(2.78±0.69) \cdot 10^2</td>
<td>(5.17±1.58) \cdot 10^2</td>
</tr>
<tr>
<td>( \eta_8 ) [A^2s]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exp.</td>
<td>(0.22±0.13) \cdot 10^{-3}</td>
<td>(0.72±0.30) \cdot 10^{-3}</td>
<td>(3.20±1.63) \cdot 10^{-3}</td>
</tr>
<tr>
<td>Sim.</td>
<td>(0.24±0.15) \cdot 10^{-3}</td>
<td>(0.73±0.28) \cdot 10^{-3}</td>
<td>(2.70±1.67) \cdot 10^{-3}</td>
</tr>
</tbody>
</table>

Table 5.2: Experimental vs. simulated microscopic performance evaluators (average and standard deviation) at low, medium and high testing condition severity.
Figure 5.18: Electric charge through the post-arc channel \([C]\) at low, medium and high testing condition severity, with average and standard deviation (white markers denote successes, black denote failures; squares denote experimental values, circles denote values computed with the LV arc black box model).

Figure 5.19: Joule’s integral \([A^2s]\) at low, medium and high testing condition severity, with average and standard deviation (white markers denote successes, black denote failures; squares denote experimental values, circles denote values computed with the LV arc black box model).
CHAPTER 5. BLACK BOX ARC MODELS

required, which is not an easy task.

5.9 Conclusions

The current chapter addresses the possibility to model the arc as a black box element of an electrical network. After a review of classical black box models, we found them not adequate to finely describe the current zero period, which experimentally proved crucial for the outcome of the interruption; see §2. We therefore tried to correct the modeling based on the physical insight gained throughout the theoretical review of arc plasma; see §3. In particular, the possible violation of the local thermal equilibrium condition for a cold plasma close to extinction at the current zero naturally lead us to introduce a non equilibrium correction for the low current regime.

The proposed model fits satisfactorily the experimental oscillograms. On the other hand, the high instability of the arc, at least in our testing conditions, results in highly scattered values of the model parameters. This is a serious obstacle for the model to be predictive. Moreover, we inherited from the classical Schwarz model the conductance dependence of the time constant and of the cooling power according to power laws. This is known to increase the fitting capability, at the expense of a clear physical interpretation of all the model parameters. For instance, the units of $\tau_0$ in (5.36) are $s \cdot \Omega^\alpha$, i.e., they depend on the value of $\alpha$.

In the end, black box modeling in low voltage proved to be descriptive. We see limited predictive possibilities when the behavior of the circuit breaker itself is concerned, e.g., during the development of a new device. On the other hand, if the behavior of a network hosting an arcing circuit breaker is concerned, then black box modeling could be of interest.
Chapter 6

Conclusions

6.1 Results

A research activity has been carried out in the framework of this thesis, in order to model, and possibly to simulate, the behavior of the electric arcs which take place during current interruptions in low voltage circuit breakers. The effort performed goes in the direction of providing the technical community of low voltage circuit breaker manufacturers with some tools for a better comprehension of such a complex phenomenon, and which are hopefully helpful in the design phase of new products. The engineering relevance of such a goal is because the electric arc in current interruption is the core technology of a circuit breaker and, at the same time, it represents the less understood and most out of control issue, which is traditionally addressed by means of trial and error experimentation. The activity we have carried out consisted of an experimental campaign, in order to provide an empirical reference, and of a theoretical analysis, in order to provide an as comprehensive as possible and self consistent explanation of the experimental evidence. Based on the knowledge acquired, we have proposed a black box approach to investigate the current zero in a descriptive manner, and we suggest to adopt a multiphysical approach to investigate the macro scale of the arc behavior in a predictive manner.

All of the experimental tests have been performed in the laboratories of ABB S.p.A. - Automation Products Division, located in Bergamo, Italy. All of the measuring instruments utilized, some of which are not standard in an industrial environment, are now currently available to ABB technicians and
will be adopted in future product validation tests. Also, suitable software routines implement the mathematical and post-processing techniques that we have adopted, so to ease technological transfer to the industrial world.

A set of performance evaluators have been proposed to judge the interruption capability of a low voltage circuit breaker during a short-circuit test. We found experimental evidence that the successfulness in current interruption is correlated with some indicators, which are based on data available in the current zero region. For such indicators (termed microscopic) a threshold value exists, telling successful interruption apart from failures. Precisely, the set of such good performance evaluators includes the arc current slope precisely at the current zero; the electric charge passing through the arc channel over a $10\mu s$-long time interval after the current zero; the Joule’s integral over a $10\mu s$-long time interval after the current zero; and finally the arc conductance precisely at current zero. On the other hand, no correlation is observed when some other evaluators (termed macroscopic) are concerned, which are based on data collected over a long time duration. The average current slope before current zero is not a good performance indicator and the theoretical reason is proved, explaining why the picture in low voltage differs from that in high-voltage. This part has been published in a journal paper [9].

Microscopic performance evaluators can be profitably used in product validating experimental campaigns, in order to gain a finer, continuously graded response than a simple, Boolean OK/KO verdict. The apparently so precious information hidden in the post-arc region requires a very sensitive measuring device and a careful post-processing of its signal. We have adopted a high sensitivity current probe and Savitsky-Golay filtering, respectively, obtaining good and stable results.

Short-circuit tests revealed a large scattering in the value of performance evaluators, even in nominally identical conditions, as regards both the breaker and the testing network. Scattering is increased in more severe testing conditions. We have come to the idea that, depending on small and unpredictable features, a low voltage circuit breaker can react in a very different way when close or above its interruption limit. Even if a success-failure threshold can still be found, the experimental evidence suggests the rather discouraging conclusion - though actually well-known - that the electric arc be a strongly stochastic phenomenon, and, as such, of difficult predictability. Careful examination of current signals close to the current zero sometimes allows sorting out anomalous failure cases, in which the circuit breaker behaved remarkably differently than in the usual case of a thermal re-strike. It is believed by the authors that current zero analysis, along with the accurate inspection of the breaker after the test, may contribute to shed some light on missed interruptions, contributing in finding a physical reason and thus a possible remedial action.
6.1. RESULTS

A fiber optics based experimental setup has been developed, in order to film the arc motion in low voltage circuit breakers under short-circuit test conditions. Optical measures can be cross-matched with electrical ones, providing an illuminating insight and a key to read testing laboratory oscillograms. The arc root motion has been studied, and the existence of a stationary time has been observed. A plausible explanation is due to the thermal inertia required by the electrodes to reach a sufficiently high temperature and emit electrons. Confrontation with electrode material melting temperatures suggests a field enhanced thermionic emission mechanism (Schottky effect). Suitably strong electric fields could be explained by means of thin space charge densities in front of the electrodes. Anyway, the arc root physics in low voltage circuit breakers is yet a non dominated issue. This part has been published in a journal paper [90].

The problem of the electric arc in low voltage circuit breakers has been theoretically set in the framework of plasma physics, in order to evidence the basic underlying principles and to provide a rigorous vision, usually not fully assessed in an industrial environment. We have provided a multi level description, each layer being of a different level of detail, trying to outline which physical features are explained by each level. We have also tried to point out the relationships of any level with the finer physical scales, in the spirit of modeling the macroscopic effects of the micro scale, so to allow recovering and accounting for what cannot be resolved at a given level of detail. The mechanism of energy transfer through collisions, especially in between particle species of similar mass, and the mechanism of energy gain from the electric field, especially for particle species of lighter mass, i.e., electrons, are pointed out as the key to understand whether the arc plasma under given conditions is in local thermal equilibrium or, rather, if non equilibrium theory should be concerned. Particularly, the arc roots and the cold plasma of an arc close to extinction seem to be the main non equilibrium dominated conditions.

Based on the relevance of the role of non equilibrium theory, a black box arc model has been proposed for low voltage circuit breakers, consisting of the classical Schwarz model for the high current regime and being suitably corrected in the low current regime. Particularly, the field correction introduced by Rieder and Urbanek is here proposed to account for the effect of the electric field in the non equilibrium plasma of the post current zero interval. A very general parameter identification method is proposed, essentially consisting in solving a constrained optimization problem by suitably coupling gradient moves with heuristic search methods. Our black box model is seen to adhere very well to experimental measures, providing remarkable improvements to the fitting capability of those models not accounting for non equilibrium physics. Such conclusions are in agreement with similar remarks made by Rieder and Urbanek.
in high-voltage, which extend also to low voltage. This part has been published in a journal paper [8], to appear.

Unfortunately, the observed, large scattering in the arc behavior, even in nominally identical conditions, is such to induce a large scattering in the optimal model parameters. As a consequence, the model can hardly be thought of as predictive in different conditions than those used for its identification. This negative conclusion prevents from using black box models to simulate the behavior of low voltage circuit breakers under testing conditions of increasing severity, until the threshold value of the performance evaluators is attained. The interrupting limit may still be computed empirically, by testing specimens of the circuit breaker at hand, in conditions of different severity, and measuring the microscopic performance evaluators. The adoption of synthetic tests produced by black box modeling was initially conceived in order to reduce the number of tests, but unfortunately this seems not possible.

In the spirit of a predictive modeling of low voltage circuit breakers, we suggest attempting a computational approach at the scale of magnetohydrodynamics (MHD). The stemming problem, which is based on the local thermal equilibrium (LTE) hypothesis, is computationally tractable on modern computers. The problem must be supplemented by suitable models to account for non equilibrium dominated features, such as the arc roots. For the latter issue, we tried to review known results in the field of high intensity discharge lamps. The output of MHD is the macroscopic scale of the arc dynamics as a conducting, viscid fluid, driven by electromagnetic forces and pressure gradients. We show that, in the typical conditions found in low voltage circuit breaker arcs, the magnetic Reynolds number is low and, consequently, weak coupling can be adopted between Navier-Stokes equations, governing the flow, and Maxwell equations, governing electromagnetism. We also propose to adopt an adequate description of the radiative heat transfer inside a participating medium, such as a possibly multi-band $P_1$ model.

The mathematical structure of the equations governing the MHD problem is briefly outlined. The numerical approximation of the stemming differential problem is tackled by means of the Galerkin method for the discretization of the spatial differential operators. Suitable and \textit{ad hoc} versions of the Galerkin method are adopted for each sub-problem, namely the Galerkin-FVM for the fluid dynamic and radiative transfer problems, the node based Galerkin-FEM for the electrostatic problem and finally the curl conform, Galerkin-edge finite element method for the magnetostatic method. Temporal differential operators are discretized by means of an implicit, second-order accurate and unconditionally stable BDF scheme. Data are passed in between sub-problems by means of a suitable distance weighted interpolation scheme. The comparison of the first qualitative results with available experiments show the feasibility of such
6.2 Recommendations for Future Research

Not surprisingly, the electric arc proved to be a tough problem, and much further research is needed before it can be dominated. The black box approach can probably be refined by increasing the complexity of the models, but the authors experience is such to discourage from doing so in a purely mathematical way, that is, by increasing the number of model parameters in search for a better fitting capability. It is believed that a real improvement can only be found if based on the physics. On the other hand, the observed, large scattering in test oscillograms under nominally identical conditions clearly witnesses the existence of physical features which are not controlled at the experimental level. Probably, the black box approach cannot be predictive, but only descriptive, when applied to low voltage circuit breakers.

With reference to currently available black box arc models, and particularly the one we propose for low voltage circuit breakers, we consider unsatisfactory the Schwarz-like conductance dependence of the time constant and of the cooling power by means of power laws. As a matter of fact, there is no physical justification for such a modeling and it is therefore difficult to obtain physically based estimates of the parameters. It could be of interest to find alternative and better functional dependencies, provided that black box models are trusted to be employed in a predictive way in circuit breaker design.

The fiber optics approach needs being intensively applied, for a better empirical comprehension of the phenomenon. Also, well designed experimental campaign should be carried out on different circuit breakers, looking for similarities and differences, as well as in order to gain statistical confidence on whatsoever conjecture. As a matter of fact, despite circuit breaker manufacturers spend a lot of testing lab time, such an activity is such not to provide a well assessed evidence. This is an unavoidable and intrinsic drawback in trial and error assisted engineering, because the product evolves along with the testing history. In the end, the duty of industrial research is to fill this gap of product development, also by testing frozen and immutable devices not for the sake of their immediate improvement, but only for the sake of knowledge and comprehension, which will ultimately result in suggestions for better products. Probably, this way can yield the greatest practical results in the industrial world.

The computational, multiphysical approach requires being deeply investigated in the next future. This is the way taken by most of worldwide manufac-
turers. Despite the overall picture is relatively well assessed, a number of very important features are still lacking a generally accepted formulation. The list includes the modeling of arc roots, of material ablation, of the initialization or arc ignition, just to cite a few examples of high impact.

A probably difficult but surely necessary experimental validation is also required for the computational results. The latter include the temperature field, which could be addressed by means of spectrometry, and the pressure field, which could be investigated by means of suitable sensors with high temporal sampling rate. Simplified cases could be conceived and numerically simulated, so to simplify the validation phase. The physical properties of the air plasma should be revised, and particularly with reference to the radiative ones, like the frequency and temperature dependence of the absorption coefficient. Most likely, a lot of know-how is already available in other branches of physics, such as the physics of the atmosphere and astronautics, and it could be profitably ported to the analysis of low voltage circuit breakers.

A rigorous mathematical analysis of the large system of PDE to be solved and of the admissibility of the numerical approach adopted should also be carried out, in order to investigate and possibly avoid hidden sources of numerical instabilities. This is by no means an easy task. As a matter of fact, Navier-Stokes equations, which are only a portion of the MHD problem, are still lacking of proofs for basic properties such as the existence of smooth solutions in the 3D case, and with bounded kinetic energy (the Navier-Stokes existence and smoothness problem [38]). Of course the numerical solution brings additional problems. Obviously, in the case of low voltage circuit breakers the only thing which is really required is a working method (and with a reasonable computational burden), the comparison with the experiment being the sole and ultimate validation. But still, some mathematics is needed to detect and fix numerical troubles, whenever they occur. Particularly, the choice of stable test spaces for the weak form, Galerkin solution should be investigated. Also, the choice of a higher order spatial discretization scheme than the finite volume method for the flow problem should be considered. The finite element method or the spectral element method could be better choices. Currently, the large availability of well assessed commercial finite volume codes is the only reason preventing from exploring other ways.
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Appendices
Appendix A

Useful Mathematical Tools

A.1 Geometrical Methods for PDE

Some classical geometric methods for partial differential equations are shortly outlined in this section. In this thesis, algebraic topology and particularly differential forms and de Rham cohomology are used to cast Maxwell equations in their potential formulation, as well as to deal with the weak form of PDE and allow their numerical solution by means of Galerkin method. The de Rham differential complex is introduced based on [15]. For a more comprehensive and canonical exposition, the reader is referenced to [63]. For an analysis-oriented exposition of cohomology, the reader is referenced to [82].

A.1.1 Cohomology of a Differential Complex

We first briefly recall some basic terminology. Let us consider a sequence \( \{C^q\}_{q \in \mathbb{Z}} \) of vector spaces and their direct sum \( C = \bigoplus_{q \in \mathbb{Z}} C^q \). Let us assume that vector space homomorphisms (i.e., linear applications) exist in between the above spaces. We shall always denote such homomorphisms as \( \partial \), even though, obviously, a different application is involved any time. Let us assume that the homomorphisms \( \partial \) be such to form the chain

\[
\ldots \xrightarrow{\partial} C^{q-1} \xrightarrow{\partial} C^q \xrightarrow{\partial} C^{q+1} \xrightarrow{\partial} \ldots \quad (A.1)
\]
Finally, let us assume that the composition of any two successive homomorphisms be the identically null homomorphism, that is,

$$\partial^2 = \partial \circ \partial = 0.$$  \hfill (A.2)

In the above hypotheses, $C$ is termed a differential complex and $\partial$ is termed the differential operator of the complex. For each homomorphism we denote its kernel (or nullspace) and its image (or range) as

$$Z^q(C) := \ker(\partial : C^q \to C^{q+1}) = \{ z \in C^q \mid \partial z = 0 \},$$

$$B^q(C) := \operatorname{im}(\partial : C^{q-1} \to C^q) = \{ b \in C^q \mid b = \partial c, c \in C^{q-1} \},$$  \hfill (A.3)

respectively. For any $q$, elements of $Z^q(C)$ are termed closed, while elements of $B^q(C)$ are termed exact. One can immediately verify that both $Z^q(C)$ and $B^q(C)$ are subspaces of $C^q$. For (A.2), any exact element is closed, so that $B^q(C) \subseteq Z^q(C)$. One can also verify that, indeed, $B^q(C)$ is a subspace of $Z^q(C)$. Therefore, the definition

$$H^q(C) := \frac{Z^q(C)}{B^q(C)},$$  \hfill (A.4)

of the quotient vector space is well posed. With reference to the algebraic structure of vectors as an Abelian, additive group, $H^q(C)$ is termed the $q$-th cohomology group of $C$. The direct sum of the cohomology groups, that is,

$$H^* (C) := \bigoplus_{q \in \mathbb{Z}} H^q(C)$$

is termed the cohomology of the differential complex. If

$$\operatorname{im}(\partial : C^{q-1} \to C^q) = \ker(\partial : C^q \to C^{q+1}),$$  \hfill (A.5)

then the sequence of vector spaces is said to be exact in $C^q$, and $B^q$ is trivially isomorphic to $Z^q$, so that $H^q = 0$, that is, the trivial group. A sequence which is exact $\forall q \in \mathbb{Z}$ is simply termed exact and its cohomology is trivial. Actually, cohomology do measure the inexactness of a differential complex.

### A.1.2 Differential Forms and de Rham Complex

Starting from this general premise, we now consider an important applicative case of differential complex. Let us consider the euclidean space $\mathbb{R}^n$, with a system of coordinates $x_1, \ldots, x_n$. (With reference to atlases, the theory is
A.1. GEOMETRICAL METHODS FOR PDE

extended to differentiable manifolds.) Let $\Omega^*$ be the algebra generated by the differentials $dx_1, \ldots, dx_n$ and subject to the relations

\[
\begin{cases}
(dx_i)^2 = 0 \\
(dx_i dx_j) = -dx_j dx_i \quad (i \neq j)
\end{cases}
\]  

(A.6)

An admissible base for the algebra $\Omega^*$ is given by

\[
1, \ dx_i, \ dx_i dx_j, \ dx_i dx_j dx_k, \ldots, \ dx_1 \ldots dx_n.
\]

(i < j)  

(i < j < k)

By using smooth functions (i.e., infinitely differentiable) $f : \mathbb{R}^n \to \mathbb{R}$ as multiplicative coefficients for the elements of the above basis, one defines

\[
\Omega^*(\mathbb{R}^n) := C^\infty(\mathbb{R}^n) \otimes \Omega^*,
\]  

(A.7)

whose elements are termed differential forms (shortly, forms) $C^\infty$ on $\mathbb{R}^n$. By construction, they can be expressed uniquely as

\[
\omega = \sum_{q \in \mathbb{N}_0} f_{i_1 \ldots i_q}(x_1, \ldots, x_n) dx_{i_1} \ldots dx_{i_q} = \sum_I f_I dx_I,
\]  

(A.8)

where $I$ is a suitable index set, introduced to provide a shorthand notation. The differential forms generated by $q$ differentials, that is, of degree (deg) $q$, are termed $q$-forms and constitute the space

\[
\Omega^q(\mathbb{R}^n) := C^\infty(\mathbb{R}^n) \otimes \mathbb{R} <dx_{i_1} \ldots dx_{i_q}>.
\]

For the sake of concreteness, we consider the $n = 3$ case, which is the one at hand in this thesis (as well as in most of the applications indeed). The 0-forms are immediately and spontaneously identified with smooth scalar fields by means of the identical isomorphism $f \mapsto f$; the 1-forms can be identified with smooth vector fields by means of the isomorphism $f_x dx + f_y dy + f_z dz \mapsto [f_x, f_y, f_z]^T$; the 2-forms can also be identified with smooth vector fields by means of the isomorphism $f_x dydz + f_y dzdx + f_z dxdy \mapsto [f_x, f_y, f_z]^T$; finally, the 3-forms can also be identified with smooth scalar fields by means of the isomorphism $f dxdydz \mapsto f$.

An exterior, or wedge, product $\wedge$ is defined on differential forms, making $\Omega^*(\mathbb{R}^n)$ an algebra. If $\tau = \sum_I f_I dx_I \in \Omega^p(\mathbb{R}^n)$ and $\omega = \sum_J g_J dx_J \in \Omega^q(\mathbb{R}^n)$ are two differential forms, their wedge product is defined as

\[
\tau \wedge \omega = \sum_{I, J} f_I g_J dx_I dx_J \in \Omega^{p+q}(\mathbb{R}^n).
\]  

(A.9)
The definitions extends straightforwardly also to generic forms in the obvious way. It is easily verified that the exterior product is not commutative, but
\[ \tau \wedge \omega = (-1)^{\text{deg } \tau \cdot \text{deg } \omega} \omega \wedge \tau. \quad (A.10) \]

There are no forms of negative degree and, owing to the first of (A.6), any form of degree greater than \( n \) is null (because at least one differential should be recurring at least twice, for the pigeonhole principle). Thus

\[ \Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^{n} \Omega^q(\mathbb{R}^n). \quad (A.11) \]

For this reason, \( \Omega^*(\mathbb{R}^n) \) is termed a graded algebra.

The exterior derivative of a differential form is inductively defined as

\[ \partial : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n) \quad (A.12) \]

with the following two rules:

if \( f \in \Omega^0(\mathbb{R}^n) \), then \( \partial f := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \in \Omega^1(\mathbb{R}^n) \)

if \( \omega = \sum f_I dx_I, \in \Omega^q(\mathbb{R}^n) \), then \( \partial \omega := \sum_I \partial f_I dx_I \in \Omega^{q+1}(\mathbb{R}^n) \)

(A.13)

For instance, in the \( n = 3 \) case, the exterior differential of 0-forms coincides with the familiar gradient operator:

\[ \partial f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz. \]

By differentiating a 1-form, a 2-form is produced and, owing to (A.6), the exterior differential of 1-forms coincides with the familiar curl operator:

\[ \partial(f_x dx + f_y dy + f_z dz) = \left( \frac{\partial f_x}{\partial x} dx + \frac{\partial f_x}{\partial y} dy + \frac{\partial f_x}{\partial z} dz \right) dx + \]
\[ \left( \frac{\partial f_y}{\partial x} dx + \frac{\partial f_y}{\partial y} dy + \frac{\partial f_y}{\partial z} dz \right) dy + \left( \frac{\partial f_z}{\partial x} dx + \frac{\partial f_z}{\partial y} dy + \frac{\partial f_z}{\partial z} dz \right) dz = \]
\[ \left( \frac{\partial f_y}{\partial y} - \frac{\partial f_y}{\partial z} \right) dy dz + \left( \frac{\partial f_z}{\partial y} - \frac{\partial f_z}{\partial x} \right) dz dx + \left( \frac{\partial f_y}{\partial z} - \frac{\partial f_y}{\partial x} \right) dxdy. \]

Similarly, by differentiating a 2-form, a 3-form is produced and, owing to (A.6), the exterior differential of 2-forms coincides with the familiar divergence oper-
\[ \partial(f_x dy dz + f_y dz dx + f_z dx dy) = \left( \frac{\partial f_x}{\partial x} dx + \frac{\partial f_x}{\partial y} dy + \frac{\partial f_x}{\partial z} dz \right) dy dz + \left( \frac{\partial f_y}{\partial x} dx + \frac{\partial f_y}{\partial y} dy + \frac{\partial f_y}{\partial z} dz \right) dz dx + \left( \frac{\partial f_z}{\partial x} dx + \frac{\partial f_z}{\partial y} dy + \frac{\partial f_z}{\partial z} dz \right) dxdy = \]
\[
\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} dxdydz.
\]

The \( \partial \) is linear by construction and thus it induces a vector space homomorphism. Moreover, it is an anti-derivative, i.e.,
\[ \partial(\tau \wedge \omega) = (\partial \tau) \wedge \omega + (-1)^{\text{deg} \tau} \tau \wedge (\partial \omega). \] (A.14)

Notice that, when applied to 0-forms, (A.14) coincides with the familiar derivation rule for the product of functions. Finally, it is immediately verified that the composition of two exterior derivatives is the identically null operator. In other terms, property (A.2) is satisfied. As a consequence
\[ \ldots \xrightarrow{\partial} \Omega^{q-1}(\mathbb{R}^n) \xrightarrow{\partial} \Omega^q(\mathbb{R}^n) \xrightarrow{\partial} \Omega^{q+1}(\mathbb{R}^n) \xrightarrow{\partial} \ldots \]

is a differential complex, termed de Rham complex.

In the \( n = 3 \) case, the de Rham complex is
\[ 0 \xrightarrow{\text{incl}} \Omega^0(\mathbb{R}^3) \xrightarrow{\text{grad}} \Omega^1(\mathbb{R}^3) \xrightarrow{\text{rot}} \Omega^2(\mathbb{R}^3) \xrightarrow{\text{div}} \Omega^3(\mathbb{R}^3) \xrightarrow{0} 0, \] (A.15)

where each instance of the polymorphic operator \( \partial \) has been explicitly indicated, step by step (the incl operator denotes the inclusion relation, rot stands for the curl, or rotor, and 0 denotes the null operator). From (A.2) the two identities \( \text{rot} \circ \text{grad} \equiv 0 \) and \( \text{div} \circ \text{rot} \equiv 0 \) follow. Translating the two identities from the original geometric terminology into an analytical one, more familiar to the literature of the applications and of differential problems, the following result immediately follows.

**Proposition A.1.** If \( \mathbf{u} \in (C^\infty(\mathbb{R}^3))^3 \) is any smooth vector field, then
\[ \nabla \times \nabla \mathbf{u} \equiv 0 \quad \text{and} \quad \nabla \cdot \nabla \times \mathbf{u} \equiv 0. \] (A.16)

### A.1.3 De Rham Cohomology Computed

Cohomology is a powerful and successful tool in algebraic topology because it is simple, and particularly because it is computable, meaning that it is simple to
compute the de Rham cohomology groups for manifolds of applicative interest. We shall deal with the very simple case of Euclidean spaces, for which de Rham cohomology groups are shortly denoted by $\mathcal{H}^q(\mathbb{R}^n)$, instead of $\mathcal{H}^q(\Omega^*(\mathbb{R}^n))$. In the $\mathbb{R}^0$ case (i.e., a topological space coinciding with a single point), the following result holds.

**Lemma A.1 (Cohomology of the point).** The only non trivial de Rham cohomology group of a space coinciding with a single point is the 0-th, which is isomorphic to $\mathbb{R}$, i.e.,

$$\mathcal{H}^0(\mathbb{R}^0) \cong \mathbb{R},$$

$$\mathcal{H}^q(\mathbb{R}^0) = 0 \quad \forall q \neq 0. \quad (A.17)$$

**Proof.** In the case at hand, the de Rham complex is $0 \rightarrow \Omega^0(\mathbb{R}^0) \rightarrow 0$ and the only a priori non trivial (for dimensional issues) cohomology group is $\mathcal{H}^0(\mathbb{R}^0)$. Nonetheless, $\mathcal{B}^0(\mathbb{R}^0) = 0$, for the inclusion map is injective and the trivial group 0 consists of just one element, and $\mathcal{Z}^0(\mathbb{R}^0) = \Omega^0(\mathbb{R}^0)$, for the null homomorphism maps anything to 0. As a consequence, $\mathcal{H}^0(\mathbb{R}^0) = \Omega^0(\mathbb{R}^0)/0 \cong \Omega^0(\mathbb{R}^0) \cong \mathbb{R}$, where the last isomorphism follows, very naturally, after identifying the functions of $\Omega^0(\mathbb{R}^0)$ with the (real) value they assume over the unique point in their domain.

We are interested in the cohomology of higher dimensional Euclidean spaces, namely $\mathbb{R}^2$ and $\mathbb{R}^3$. To compute it, we start noticing that $\Omega^*$ is a contravariant functor from the category of Euclidean spaces $\{\mathbb{R}^n\}_{n \in \mathbb{Z}}$ and their smooth maps to the category of graded algebras $\{\mathcal{H}^*(\mathbb{R}^n)\}_{n \in \mathbb{Z}}$ and their homomorphisms. Precisely, a smooth map

$$f : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

between Euclidean spaces is mapped by the functor $\Omega^*$ to the composite map (or pullback)

$$\Omega^*(f) := f^* : \Omega^*(\mathbb{R}^n) \rightarrow \Omega^*(\mathbb{R}^m),$$

defined so as that

$$\omega \mapsto f^*(\omega) := \omega \circ f$$

holds for any differential form $\omega$ on $\mathbb{R}^n$. The functional sense of the pullback, and particularly the reason for its name (i.e., a form defined over the image manifold of the map $f$ is pulled back to a form defined over the domain manifold), is shown in the following diagram.

\[
\begin{array}{ccc}
\mathbb{R}^m & \xrightarrow{f} & \mathbb{R}^n \\
\downarrow f^*(\omega) := \omega \circ f & & \downarrow \omega \\
\end{array}
\]
The definition of pullback actually makes $\Omega^*$ a contravariant functor (that is, it conserves the identity map and the composition of maps, in reverse order), owing to the following

**Proposition A.2** (Functor properties of $\Omega^*$). Let $id_{\mathbb{R}^n}$ and $id_{\Omega^*(\mathbb{R}^n)}$ be the identity maps on $\mathbb{R}^n$ and $\Omega^*(\mathbb{R}^n)$, respectively. Let $f : \mathbb{R}^l \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$, $l,m,n \in \mathbb{N}_0$, be any two maps. Then

$$id_{\mathbb{R}^n} = \Omega^*(id_{\mathbb{R}^n}) = id_{\Omega^*(\mathbb{R}^n)},$$

$$(g \circ f)^* = \Omega^*(g \circ f) = \Omega^*(f) \circ \Omega^*(g) = f^* \circ g^*.$$

**Proof.** For a better understanding of both the theorem and its proof, the following two diagrams may be helpful.

![Diagram](image)

By definition of pullback, if $\omega$ is a generic differential form on $\mathbb{R}^n$, then $id_{\mathbb{R}^n}^*(\omega) = \omega \circ id_{\mathbb{R}^n} = \omega$ (see left diagram) and $(g \circ f)^*(\omega) = \omega \circ (g \circ f) = (\omega \circ g) \circ f = g^*(\omega) \circ f = f^*(g^*(\omega)) = (f^* \circ g^*)(\omega)$ (see right diagram).

In order to compute the cohomology of more complex topological spaces than the single point, we need the other classical tool in algebraic topology, namely homotopy [55]. We recall that, given two topological spaces $X$ and $Y$, two continuous functions $f_0, f_1 : X \to Y$ are homotopic, indicated $f_0 \sim f_1$, iff a continuous function $F : X \times [0,1] \to Y$ exists, such that $F(x,0) \equiv f_0(x)$ and $F(x,1) \equiv f_1(x)$, $\forall x \in X$, meaning that the two functions may be continuously deformed one into the other. Also, two topological spaces $X$ and $Y$ are homotopic iff two continuous functions $f : X \to Y$ and $g : Y \to X$ exist, such that $g \circ f \sim id_X$ and $f \circ g \sim id_Y$. We denote by the symbol $\sim$ the homotopy relation between both homotopic maps and homotopic topological spaces. Clearly, function and space homotopy are equivalence relations and thus they are transitive. The following obvious result holds.

**Proposition A.3.** Euclidean spaces are homotopic, i.e.,

$$\mathbb{R}^{n+1} \sim \mathbb{R}^n \sim \ldots \sim \mathbb{R}^0.$$

**Proof.** Let us consider $\mathbb{R}^n$, with coordinates $x_1, \ldots, x_n$, and $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$, with coordinates $x_1, \ldots, x_n, x_{n+1}$. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection, defined as $(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n)$ and $s : \mathbb{R}^n \to \mathbb{R}^{n+1}$ the inclusion map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. One may immediately verify that $\pi \circ s = id$ and that $s \circ \pi \sim id$. Therefore, $\mathbb{R}^{n+1} \sim \mathbb{R}^n$, whence proposition A.3 follows inductively. \qed
APPENDIX A. USEFUL MATHEMATICAL TOOLS

The bridge between homotopy and cohomology consists in the property of homotopic spaces to have the same cohomology. The following theorem illustrates the case of Euclidean spaces.

**Theorem A.1.** Euclidean spaces have the same de Rham cohomology, i.e.,

\[
\mathcal{H}^\ast(\mathbb{R}^{n+1}) \cong \mathcal{H}^\ast(\mathbb{R}^n) \cong \ldots \cong \mathcal{H}^\ast(\mathbb{R}^0).
\] (A.18)

**Proof.** We introduce the homotopy operator \( K : \Omega^\ast(\mathbb{R}^n \oplus \mathbb{R}) \to \Omega^\ast(\mathbb{R}^n \oplus \mathbb{R}) \), defined by

\[
(\pi^\ast \omega)(f(x_1, \ldots, x_n, x_{n+1}) \mapsto 0
\]

and by

\[
(\pi^\ast \omega)(f(x_1, \ldots, x_n, x_{n+1})dx_{n+1} \mapsto \int_0^{x_{n+1}} f(x_1, \ldots, x_n, \xi) \, d\xi,
\]

for any possible differential form \( \omega \in \Omega^\ast(\mathbb{R}^n) \) and any possible smooth map \( f : \mathbb{R}^{n+1} \to \mathbb{R} \) (it is easily understood that any form on \( \Omega^\ast(\mathbb{R}^{n+1}) \) is a linear combination of one of the two possibilities above, on which \( K \) has been defined). Let \( id \) be the generic identity map, without explicitly mentioning on which space, which is clear from the context. With simple computations, one may prove the identity

\[
(id - \pi^\ast \circ s^\ast)(\omega) \equiv (\partial \circ K - K \circ \partial)(\omega), \quad \forall \omega \in \Omega^\ast(\mathbb{R}^n).
\]

In the special case of closed forms \( \partial \omega = 0 \), one gets

\[
(id - \pi^\ast \circ s^\ast)(\omega) = \partial(K\omega),
\]

that is, an exact form is obtained, which consequently falls in the equivalence class of the zero of \( \mathcal{H}^\ast(\mathbb{R}^n \oplus \mathbb{R}) \). In other terms, \( \pi^\ast \circ s^\ast = id \) in the cohomology, that means, they fall in the same equivalence class, or they are the same element in the cohomology group. For the functor properties of \( \Omega^\ast \), from \( \pi \circ s = id \) one immediately gets \( s^\ast \circ \pi^\ast = id^\ast = id \), so that \( \mathcal{H}^\ast(\mathbb{R}^{n+1} \oplus \mathbb{R}) \) and \( \mathcal{H}^\ast(\mathbb{R}^n) \) are isomorphic. The other equalities of theorem A.1 follow inductively.

The cohomology of euclidean spaces thus follows straightforwardly as a corollary of theorem A.1 and of the cohomology of the point.

**Corollary A.1** (Cohomology of Euclidean spaces). The only non trivial de Rham cohomology group of an Euclidean space \( \mathbb{R}^n \) is the 0-th, which is isomorphic to \( \mathbb{R} \), i.e.,

\[
\mathcal{H}^0(\mathbb{R}^n) \cong \mathbb{R},
\]

\[
\mathcal{H}^q(\mathbb{R}^n) = 0 \quad \forall q \neq 0.
\] (A.19)

Particularly, the first and second de Rham cohomology group are trivial. Therefore, the de Rham complex of an Euclidean space is an exact sequence in \( \Omega^1(\mathbb{R}^n) \) and in \( \Omega^2(\mathbb{R}^n) \). As a consequence, two corollaries follow immediately, which allow casting the electrostatic and the magnetostatic problem into their potential form, thus allowing their numerical solution by means of the Galerkin-FEM method. When enunciating the following two corollaries, we use an analytical terminology, which is closer to that of the applications, instead of a geometrical one.
Corollary A.2 (Scalar potential). Let \( u \in (C^\infty(\mathbb{R}^3))^3 \) be any curl-free (i.e., irrotational), smooth vector field on \( \mathbb{R}^3 \). Then \( u \) is the gradient of some suitable smooth scalar field \( f \in C^\infty(\mathbb{R}^3) \) (termed scalar potential), i.e.,

\[
\nabla \times u = 0 \implies \exists f \in C^\infty(\mathbb{R}^3) \mid u = \nabla f.
\]

(A.20)

The scalar potential is not defined uniquely, for an equally admissible scalar potential is obtained by summing to \( f \) any constant.

Proof. \( H^1(\mathbb{R}^3) = 0 \) implies that all the closed 1-forms (i.e., curl-free) are exact. Then the property (A.20) immediately follows from the meaning of the gradient operator as the instance assumed by the exterior differential \( \partial \) from \( \Omega^0(\mathbb{R}^3) \) to \( \Omega^1(\mathbb{R}^3) \). The non uniqueness of the scalar potential follows immediately because constants do coincide with closed 0-forms, that is, with the nullspace of the gradient; see the chain (A.15) and account for the property (A.2).

Corollary A.3 (Vector potential). Let \( u \in (C^\infty(\mathbb{R}^3))^3 \) be a divergence-free (i.e., solenoidal), smooth vector field on \( \mathbb{R}^3 \). Then \( u \) is the curl of some suitable smooth vector field \( v \in (C^\infty(\mathbb{R}^3))^3 \) (termed vector potential), i.e.,

\[
\nabla \cdot u = 0 \implies \exists v \in (C^\infty(\mathbb{R}^3))^3 \mid u = \nabla \times v.
\]

(A.21)

The vector potential is not defined uniquely, for an equally admissible vector potential is obtained by summing to \( v \) any gradient.

Proof. \( H^2(\mathbb{R}^3) = 0 \) implies that all the closed 2-forms (i.e., divergence-free) are exact. Then the property (A.21) immediately follows from the meaning of the curl operator as the instance assumed by the exterior differential \( \partial \) from \( \Omega^1(\mathbb{R}^3) \) to \( \Omega^2(\mathbb{R}^3) \). The non uniqueness of the vector potential follows immediately because, for \( H^1(\mathbb{R}^3) = 0 \), gradients, that is, exact 1-forms, do coincide with closed 2-forms, that is, with the nullspace of the curl; see the chain (A.15) and account for the property (A.2), or simply recall the proposition A.1.

Remark. The above results, based on the cohomology of \( \mathbb{R}^n \), also hold in the case of topological spaces which are homotopic to Euclidean spaces and particularly, since Euclidean spaces are homotopic to a point, to null-homotopic, or contractible spaces. More generally, homotopic topological spaces have isomorphic cohomology groups. This is the celebrated Poincaré’s lemma. A special and very useful case is represented by bounded, contractible domains \( \Omega \subset \mathbb{R}^n \), and particularly in the \( n = 3 \) case. In the case of domains with internal cavities or traversed by tunnels, the cohomology is not so trivial and the potentials assume a more intriguing form. We shall never consider this possibility in this thesis, anyway.
A.2 Stokes’ Theorem and Its Applications

On differential forms (see §A.1) it is possible to develop an integral calculus theory [63]. The fundamental result is a celebrated theorem from Stokes of stunning depth, simplicity, formal symmetry, conciseness and beauty.

**Theorem A.2** (Stokes’ Theorem). Let \( \omega \) be a \( n \)-form, \( \Omega \) be a smooth, orientable, \((n+1)\)-manifold and \( \partial \Omega \) be its boundary, with the induced orientation. Then

\[
\int_{\Omega} \partial \omega = \int_{\partial \Omega} \omega. \tag{A.22}
\]

Notice that differential forms are integrated over manifolds of the same dimension: \( \omega \) is an \( n \)-form and is integrated over the \( n \)-manifold \( \partial \Omega \), while \( \partial \omega \) is an \((n+1)\)-form and is integrated over the \((n+1)\)-manifold \( \Omega \). Stokes’ theorem is not only theoretically elegant, it is also a powerful tool rich of applications. We have used many corollaries of Stokes’ theorem which extend the familiar integration by parts rule to multivariable calculus in multidimensional spaces. We shall always assume in what follows to be dealing with “sufficiently regular”, or Lipschitz [31], domains (seen as special case of manifolds), which is frequently the case in many physical and engineering applications.

A.2.1 The Curl Theorem

The familiar curl theorem follows immediately as a corollary of Stokes’ theorem for a 1-form \( \omega \). It states that the flux of the curl of a vector field over a surface is equal to the circulation of the vector field itself over the boundary of the surface. Formally, we have the following

**Corollary A.4** (Curl Theorem). Let \( \Omega \subset \mathbb{R}^3 \) be a bounded, orientable surface, \( \Gamma := \partial \Omega \) be its boundary, with positive orientation, \( N(\Omega) \) be the normal bundle of \( \Omega \), \( \mathbf{n} \in N(\Omega) \) be the unit normal to \( \Omega \), \( T(\Gamma) \) be the tangent bundle of \( \Gamma \), \( \mathbf{t} \in T(\Gamma) \) be the unit tangent to \( \Gamma \) and \( \mathbf{v} : \Omega \to \mathbb{R}^3 \) be a smooth vector field. Then

\[
\int_{\Omega} (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, d\Omega = \int_{\partial \Omega} \mathbf{v} \cdot \mathbf{t} \, d\Gamma. \tag{A.23}
\]

**Proof.** For the sake of brevity, we introduce the convention that \( i, j, k \) be an even index
permutation $\sigma$ of 1, 2, 3, such that $j = \sigma(i)$, $k = \sigma(j)$ and thus $i = \sigma(k)$. This said, one has

$$\int_{\partial \Omega} v \cdot t \, d\Gamma = \int_{\partial \Omega} \sum_i v_i \, dx_i$$

$$= \int_{\Omega} \sum_i (\partial_j v_i \cdot dx_j \wedge dx_i)$$

$$= \int_{\Omega} \sum_i (\partial_j v_k - \partial_k v_j) \cdot dx_i \wedge dx_j \wedge dx_k$$

$$= \int_{\Omega} (\nabla \times v) \cdot n \, d\Omega$$

where the first step is simply the transcription of the circulation of $v$ over $\partial \Omega$ in the language of differential forms, then Stokes’ theorem is applied and the exterior derivative is computed, summations are rearranged (the $\wedge$ symbol denotes a missing differential) and finally the result is translated back in the language of analysis, yielding the flux of the curl of $v$.

A.2.2 The Divergence Theorem

The familiar divergence theorem follows immediately as a corollary of Stokes’ theorem for a 2-form $\omega$. It states that the integral of the divergence of a vector field over a bounded domain is equal to the flux of the vector field itself over the boundary of the domain. Formally, we have the following

**Corollary A.5** (Divergence Theorem). Let $\Omega \subset \mathbb{R}^3$ be a bounded, Lipschitz domain, $\Gamma := \partial \Omega$ be its boundary, $N(\Gamma)$ be the normal bundle of $\Gamma$, $n \in N(\Gamma)$ be the unit, outward normal and $v : \Omega \to \mathbb{R}^3$ be a smooth vector field. Then

$$\int_{\Omega} \nabla \cdot v \, d\Omega = \int_{\partial \Omega} v \cdot n \, d\Gamma. \quad (A.24)$$

**Proof.** For the sake of brevity, we introduce the convention that $i, j, k$ be an even index permutation $\sigma$ of 1, 2, 3, such that $j = \sigma(i)$, $k = \sigma(j)$ and thus $i = \sigma(k)$. This said, one has

$$\int_{\partial \Omega} v \cdot n \, d\Gamma = \int_{\partial \Omega} \sum_i v_i \cdot dx_j \wedge dx_k = \int_{\Omega} \sum_i \partial_i v_i \cdot dx_i \wedge dx_j \wedge dx_k = \int_{\Omega} \nabla \cdot v \, d\Omega,$$

where the first step is simply the transcription of the flux of $v$ over $\partial \Omega$ in the language of differential forms, then Stokes’ theorem is applied and the exterior derivative is computed, and finally the result is translated back in the language of analysis, yielding the divergence of $v$. \qed

A.2.3 Green’s Identity

The theorem of divergence allows deducing a useful tool for when dealing with the weak form of PDE, namely the

**Lemma A.2** (Green’s identity). Let $\Omega \subset \mathbb{R}^3$ be a bounded, Lipschitz domain, $\Gamma := \partial \Omega$ be its boundary, $N(\Gamma)$ be the normal bundle of $\Gamma$, $n \in N(\Gamma)$ be the
unit, outward normal and \( \varphi, \psi : \Omega \to \mathbb{R} \) be two smooth scalar fields. Then

$$
\int_{\Omega} (\partial_i \varphi) \cdot \psi \, d\Omega = -\int_{\Omega} \varphi \cdot (\partial_i \psi) \, d\Omega + \int_{\partial \Omega} \varphi \cdot \psi \cdot n_i \, d\Gamma. \tag{A.25}
$$

**Proof.** Lemma A.2 follows immediately after the divergence theorem A.5, when the special function \( v := \varphi \cdot \psi \) is concerned.

In the next two sections we shall show how to use Green’s identity to modify the weak form of a PDE, by transferring part of the derivatives involved in the governing differential operator from the sought solution to the test function. We shall regard the two operators involved in the electrostatic and magnetostatic problems, namely the Laplacian and the curl-curl, respectively.

### A.2.4 Green’s Formula for the Laplacian Operator

**Theorem A.3** (Green’s formula for the Laplacian operator). Let \( \Omega \subset \mathbb{R}^3 \) be a bounded, Lipschitz domain, \( \Gamma := \partial \Omega \) be its boundary, \( N(\Gamma) \) be the normal bundle of \( \Gamma \), \( \mathbf{n} \in N(\Gamma) \) be the unit, outward normal and \( \kappa, u, v : \Omega \to \mathbb{R} \) be three smooth scalar fields. Then

$$
\int_{\Omega} (\nabla \cdot \kappa \nabla u) \cdot v \, d\Omega = -\int_{\Omega} \kappa \nabla u \cdot \nabla v \, d\Omega + \int_{\partial \Omega} \kappa \cdot \partial_n u \cdot v \, d\Gamma. \tag{A.26}
$$

**Proof.** Lemma A.2, with reference to the special functions \( \varphi := \kappa \partial_i u \) and \( \psi := v \), reads

$$
\int_{\Omega} (\partial_i (\kappa \partial_i u)) \cdot v \, d\Omega = -\int_{\Omega} \kappa \partial_i u \cdot \partial_i v \, d\Omega + \int_{\partial \Omega} \kappa \partial_i u \cdot n_i \cdot v \, d\Gamma.
$$

Summing over the three vector components, that is, for \( i \in \{1, 2, 3\} \), and recalling the gradient theorem \( \mathbf{n} \cdot \nabla u = \partial_n u \), then theorem A.3 immediately follows.

### A.2.5 Green’s Formula for the curl-curl Operator

**Theorem A.4** (Green’s formula for the curl-curl operator). Let \( \Omega \subset \mathbb{R}^3 \) be a bounded, Lipschitz domain, \( \Gamma := \partial \Omega \) be its boundary, \( N(\Gamma) \) be the normal bundle of \( \Gamma \), \( \mathbf{n} \in N(\Gamma) \) be the unit, outward normal, \( \mathbf{u}, \mathbf{v} : \Omega \to \mathbb{R}^3 \) be two smooth vector fields and \( \kappa : \Omega \to \mathbb{R} \) be a smooth scalar field. Then

$$
\int_{\Omega} (\nabla \times \kappa \nabla \times \mathbf{u}) \cdot \mathbf{v} \, d\Omega = \int_{\Omega} (\kappa \nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \, d\Omega + \int_{\partial \Omega} (\mathbf{n} \times \kappa \nabla \times \mathbf{u}) \cdot \mathbf{v} \, d\Gamma. \tag{A.27}
$$
Proof. For the sake of brevity, we introduce the convention that $i, j, k$ be an even index permutation $\sigma$ of 1, 2, 3, such that $j = \sigma(i)$, $k = \sigma(j)$ and thus $i = \sigma(k)$. For the sake of simplicity, we pass through the extrinsic choice to work in orthogonal Cartesian coordinates and, afterward, we recover the intrinsic operatorial form. By applying Lemma A.2 twice and by rearranging the summations over the index set, one gets the sequence of identities

\[
\int_{\Omega} (\nabla \times w) \cdot v \ d\Omega = \int_{\Omega} \sum_i (\partial_j w_k - \partial_k w_j) v_i \ d\Omega
\]

\[
= \sum_i \int_{\Omega} (\partial_j w_k) v_i \ d\Omega - \sum_i \int_{\Omega} (\partial_k w_j) v_i \ d\Omega
\]

\[
= -\sum_i \int_{\Omega} w_k (\partial_j v_i) \ d\Omega + \sum_i \int_{\partial \Omega} w_k v_i n_j \ d\Gamma
\]

\[
+ \sum_i \int_{\Omega} w_j (\partial_k v_i) \ d\Omega - \sum_i \int_{\partial \Omega} w_j v_i n_k \ d\Gamma
\]

\[
= \int_{\Omega} \sum_i w_i (\partial_j v_k - \partial_k v_j) \ d\Omega + \int_{\partial \Omega} (n_j w_k - n_k w_j) v_i \ d\Gamma
\]

\[
= \int_{\Omega} w \cdot (\nabla \times v) \ d\Omega + \int_{\partial \Omega} (n \times w) \cdot v \ d\Gamma,
\]

whence theorem A.4 follows as a special case, by setting $w := \kappa \nabla \times u$. \hfill \qed

A.3 Geometry of the 2-Sphere

A.3.1 Spherical Coordinates

In the treatment of radiation we made use of a spherical coordinate system, as indicated in Figure A.1, where $r \geq 0$ is the distance from the origin, $\theta \in [0, \pi]$ is the colatitude and $\phi \in [0, 2\pi]$ is the longitude. Since we deal with solid
angles, it is convenient to refer to the unit 2-sphere $S^2$, which is the 2-manifold embedded in $\mathbb{R}^3$ satisfying the constraint $|s| = 1$ (a fact that we explicitly emphasize by writing $\hat{s}$, with the $\hat{\cdot}$ sign). The latter equation in the spherical coordinate system $(r, \theta, \phi)$ accounts to $r = 1$. The points of $S^2$ are in bijective correspondence with the unit vectors $\hat{s} \in \mathbb{R}^3$ and thus we identify the two entities and write $\hat{s} \in S^2$. In cartesian coordinates,

\[ s = r \sin \theta \cos \phi \cdot \hat{e}_1 + r \sin \theta \sin \phi \cdot \hat{e}_2 + r \cos \theta \cdot \hat{e}_3. \]  
(A.28)

and, particularly,

\[ \hat{s} = \sin \theta \cos \phi \cdot \hat{e}_1 + \sin \theta \sin \phi \cdot \hat{e}_2 + \cos \theta \cdot \hat{e}_3. \]  
(A.29)

## A.3.2 Spherical Harmonics

Spherical harmonics are an infinite set of functions $Y^m_l : S^2 \to \mathbb{C}$, for $l \in \mathbb{N}_0$ and $m \in \{-l, \ldots, +l\}$, defined over the 2-sphere $S^2$. The explicit expression of spherical harmonics may be found on many treatises [60, 81] and reads

\[ Y^m_l(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!}} P^m_l(\cos \theta) e^{im\phi}, \]  
(A.30)

where $P^m_l(\xi)$, for $l \in \mathbb{N}_0$ and $m \in \{-l, \ldots, +l\}$, are the associated Legendre functions. The latter are defined as the canonical solutions of the generalized Legendre equation

\[ (1 - \xi^2) \eta''(\xi) - 2\xi \eta'(\xi) + \left( l(l+1) - \frac{m^2}{1 - \xi^2} \right) \eta(\xi) = 0, \]

and may be explicitly computed [1] as

\[ P^m_l(\xi) = (-1)^m (1 - \xi^2)^{m/2} \frac{d^m}{d\xi^m} P_l(\xi), \]

where $P_l(\xi)$, for $l \in \mathbb{N}_0$, are Legendre polynomials

\[ P_l(\xi) = \frac{1}{2^l l!} \frac{d^l}{d\xi^l} (\xi^2 - 1)^l. \]

Spherical harmonics constitute a set of orthogonal functions and the expression under square root on the r.h.s. of (A.30) is such to normalize them over the 2-sphere, according to the $L^2$ norm, that is,

\[ (Y^m_1, Y^m_2)_{L^2(S^2)} = \int_{S^2} Y^{m_1}_1 \overline{Y^{m_2}_2} \hat{s} = \delta_{l_1 l_2} \delta_{m_1 m_2}, \]
where $\delta$ is the Kronecker symbol. For this reason, we use spherical harmonics to expand square summable functions over the 2-sphere in a (truncated) Fourier series, e.g., when dealing with $P_N$ methods for discretizing radiance over the solid angle. The main idea is to use the decomposition

$$L^2(S^2) = \bigoplus_{l=0}^{\infty} \text{span}\{Y^m_l\}_{m=-l}^l.$$

Actually, we find it computationally more simple not to use directly spherical harmonics, but to use a different expression, in terms of polynomials, for $\text{span}\{Y^m_l\}_{m=-l}$, $\forall l \in \mathbb{N}_0$. Particularly, we will use the decomposition

$$L^2(S^2) = \bigoplus_{l=0}^{\infty} V_l,$$

where

$$V_l \cong \text{span}\{Y^m_l\}_{m=-l}, \quad \forall l \in \mathbb{N}_0,$$

is obtained as a vector space of polynomials of degree $l$ and satisfying the characterizing property of spherical harmonics of being the angular part of the functions $f$ satisfying Laplace equation

$$\nabla^2 f = 0. \quad (A.31)$$

It is convenient to work in Cartesian coordinates $(\hat{s}_1, \hat{s}_2, \hat{s}_3)^1$ and then return to spherical coordinates by means of (A.28). We systematically look for solutions of (A.31) in polynomial form and express $f \in \mathbb{R}[\hat{s}_1, \hat{s}_2, \hat{s}_3]$ as the sum $f = \sum_{l \in \mathbb{N}_0} f_l$ of polynomials $f_l \in \mathbb{R}[\hat{s}_1, \hat{s}_2, \hat{s}_3]$ of degree $l$. For $l = 0$ one gets the constant function $f_0(\hat{s}_1, \hat{s}_2, \hat{s}_3) = a^{(0)}$, which is immediately seen to satisfy Laplace equation (A.31). The first subspace found has dimension 1, as expected, and reads

$$V_0 := \{a^{(0)} \cdot 1 \mid a^{(0)} \in \mathbb{R}\}.$$

Likewise, for $l = 1$ one gets $f_1(\hat{s}_1, \hat{s}_2, \hat{s}_3) = \sum_{i=1}^{3} a^{(1)}_i \hat{s}_i$, which is also a solution to (A.31) for any combination of the constant parameters $a^{(1)}_i$. The latter can be collected into a rank 1 tensor (i.e., a vector). The second subspace found has dimension 3, as expected, and reads

$$V_1 := \left\{ \sum_{i=1}^{3} a^{(1)}_i \hat{s}_i \mid a^{(1)}_i \in \mathbb{R} \right\}.$$

---

1We purposely use $\hat{s}$ and not $x$ to avoid confusion in between position (i.e., the latter) and direction (i.e., the former) in $\mathbb{R}^3$ and to stick to the notation adopted when handling radiation theory.
For $l = 2$ one gets $f_2(\hat{s}_1, \hat{s}_2, \hat{s}_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}^{(2)} \hat{s}_i \hat{s}_j$. In this case, Laplace equation (A.31) imposes the constraint

$$0 = \sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial^2}{\partial \hat{s}_k^2} \sum_{i=1}^{3} a_{ij}^{(2)} \hat{s}_i \hat{s}_j = \sum_{i=1}^{3} a_{ii}^{(2)}.$$ 

Therefore the coefficients $a_{ij}^{(2)}$ may be collected into a traceless rank 2 tensor (i.e., a traceless matrix). Without any loss of generality, one can always assume that such a tensor be symmetric, for one could always set $a_{ij}^{(2)} := a_{ji}^{(2)} := (a_{ij}^{(2)} + a_{ji}^{(2)})/2$, i.e., one can always consider the symmetric part of the matrix without modifying the relevant quadratic form. (An equivalent way to see this is to decompose the matrix into its symmetric and antisymmetric component and then to notice that any antisymmetric matrix $M$ always gives rise to an identically vanishing quadratic form, for $s^T Ms = (s^T Ms)^T = s^T M^T s = -s^T Ms \Rightarrow s^T Ms = 0$.) Summing up, matrix $a_{ij}^{(2)}$ has 9 entries, symmetry fixes 3 of which and tracelessness adds one more linear constraint, so that the third subspace found has dimension 5, as expected, and reads

$$V_2 := \left\{ \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}^{(2)} \hat{s}_i \hat{s}_j \Bigg| \sum_{i=1}^{3} a_{ii}^{(2)} = 0, a_{ij}^{(2)} = a_{ji}^{(2)} \ \forall i,j \right\},$$

One can also advance further to the higher dimensional subspaces $V_l$, collecting coefficients in traceless (meaning that all the traces are null, for any two indexes) and symmetric, rank $l$ tensors $a_{i_1...i_l}^{(l)}$.

Since in spherical coordinates $(r, \theta, \phi)$ the Laplacian (in $\mathbb{R}^3$) reads

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2},$$

one understands that

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \nabla^2_{S^2}, \quad (A.32)$$

where $\nabla^2_{S^2}$ is the restriction of Laplace operator over the 2-sphere. By construction, any member of $V_l$ is a homogeneous polynomial of degree $l$ which, when expressed in spherical coordinates by means of (A.28), takes the form of a function of type $f(r, \theta, \phi) = r^lg(\theta, \phi)$. Plugging into (A.32) and rearranging, one gets

$$- \nabla^2_{S^2} g(\theta, \phi) = l(l+1)g(\theta, \phi), \quad (A.33)$$

showing that the angular portion of any member of $V_l$ is an eigenfunction of the restriction of Laplace operator over the 2-sphere, relevant to the eigenvalue.
A.3. GEOMETRY OF THE 2-SPHERE

Since the Laplacian is a self-adjoint operator, eigenspaces relevant to different eigenvalues are mutually orthogonal, and therefore \( V_{l_1} \perp V_{l_2} \), for \( l_1 \neq l_2 \), or \( V_{l_1} \cap V_{l_2} = \{0\} \). This can also help noticeably when computing the integrals involving polynomials in the components of \( \hat{s} \) required in developing \( P_N \) models for radiation.

A.3.3 Useful Integrals

Some definite integrals are used throughout radiation theory, which we hereafter compute once and for all. We remind that \( d\hat{s} = \sin \theta \, d\phi \, d\theta \). If one wants to compute integrals explicitly (which could take a long time!) all that is needed in radiation theory is

\[
\int_0^\pi d\theta \int_0^{2\pi} d\phi \, \sin \theta \cdot \hat{s}^{a_i}(\theta, \phi) \cdot \hat{s}^{a_j}(\theta, \phi) \cdot \hat{s}^{a_k}(\theta, \phi),
\]

for different (and possibly null) integer exponents \( a_i, a_j, a_k \). To make computations is an error prone and not particularly smart practice, so we will take shortcuts, whenever possible. In the followings we shall use extensively the fact that any component of \( \hat{s} \) is equivalent to the other two, by symmetry under rotation of the 2-sphere.

Lemma A.3. For any \( i \in \{1, 2, 3\} \),

\[
\int_{S^2} \hat{s}^{2i} \, d\hat{s} = \frac{4\pi}{3}.
\]  

(A.34)

Proof. One may set \( J := \int s^{2i} \, d\hat{s} \) and argument that \( 3J = \sum_{i=1}^3 \int s^2 \, d\hat{s} = \int \sum_{i=1}^3 s^2 \, d\hat{s} = \int 1 \, d\hat{s} = 4\pi \), from which \( J = 4\pi/3 \). \( \square \)

The previous result is a special case of

Lemma A.4. For any \( i \in \{1, 2, 3\} \) and for any integer \( n \geq 0 \),

\[
\int_{S^2} \hat{s}^{2ni} \, d\hat{s} = \frac{4\pi}{2n+1}.
\]  

(A.35)

Proof. Symmetry allows choosing any value for \( i \). It is convenient to choose \( i = 3 \), i.e., \( \hat{s}_3 = \cos \theta \) and compute the elementary integral

\[
\int_0^\pi d\theta \int_0^{2\pi} d\phi \, \sin \theta (\cos \theta)^{2n} = 2\pi \left[ \frac{(\cos \theta)^{2n+1}}{2n+1} \right]_0^\pi,
\]

whence the result. \( \square \)
Lemma A.5. For any two distinct \( i, j \in \{1, 2, 3\} \),

\[
\int_{\mathbb{S}^2} \hat{s}_i \hat{s}_j \, d\hat{s} = 0.
\] (A.36)

Proof. One may view the integral as an inner product in \( L^2(\mathbb{S}^2) \), namely \( \int 1 \cdot s_i s_j \, d\hat{s} = (1, s_i s_j) \), with \( 1 \in V_0 \) and \( s_i s_j \in V_2 \), reminding that \( V_0 \) and \( V_2 \) are eigenspaces relevant to different eigenvalues and thus mutually orthogonal. That \( s_i s_j \in V_2 \) one may understand since the rank 2 tensor associated to such a quadratic form has null entries with the exception of \( a^{(2)}_{ij} = a^{(2)}_{ji} = 1/2 \), and it is thus traceless and symmetric.

The previous result falls into a wider one. Whenever at least one exponent \( a_i, a_j, a_k \), say \( a_i \), is odd, the relevant integral vanishes, for one could partition the 2-sphere into two portions, with conjugate points over the two components having opposite \( \hat{s}_i \) and same \( \hat{s}_j \) and \( \hat{s}_k \). The contributions of conjugate points annihilate. This is very useful in radiation theory, for it allows reducing a lot the computations required to get the moments of spectral radiance expressed in terms of spherical harmonics (see §3.8.3). To the extent required by \( P_1 \) and \( P_2 \) methods, the only non vanishing integral still to be explicitly computed is covered by the following

Lemma A.6. For any two distinct \( i, j \in \{1, 2, 3\} \),

\[
\int_{\mathbb{S}^2} \hat{s}_i^2 \hat{s}_j^2 \, d\hat{s} = \frac{4\pi}{15}.
\] (A.37)

Proof. One may set \( J := \int s_i^2 s_j^2 \, d\hat{s} \) and \( K := \int s_i^4 \, d\hat{s} = 4\pi/5 \) (by Lemma A.4, for \( n = 2 \)) and argument that

\[
4\pi = \int_{\mathbb{S}^2} \left( \sum_{i=1}^{3} s_i^2 \right)^2 \, d\hat{s} = 3K + 6J,
\]

whence the result. 

We close this section with some integrals required to compute the first moments of Boltzmann equation for photons (see §3.8.2).

Proposition A.4.

\[
\int_{\mathbb{S}^2} d\hat{s} = 4\pi.
\] (A.38)

Proof. The result follows immediately if one recalls the sphere surface formula. Alternatively, one may carry out the integration explicitly, from which the sphere surface formula derives.
Proposition A.5. 
\[ \int_{\mathbb{S}^2} \hat{s} \, d\hat{s} = 0. \] (A.39)

Proof. The result follows from noticing that couples of antipodal points over the sphere yield opposite contribution and thus annihilate. Another smart way to proceed is to notice that \( 1 \in V_0 \) and \( \hat{s}_i \in V_1 \), so that they belong to mutually orthogonal subspaces. \( \Box \)

Proposition A.6. 
\[ \int_{\mathbb{S}^2} \hat{s} \otimes \hat{s} \, d\hat{s} = \frac{4\pi}{3} I. \] (A.40)

Proof. The case \( \hat{s}_i^2 \) and \( \hat{s}_i \hat{s}_j \) are covered by Lemma A.3 and A.5, respectively. \( \Box \)

A.3.4 The Half Sphere

When dealing radiative boundary conditions, only half space and half sphere is of interest. If one defines a spherical coordinate system lined up with the normal to the boundary, as indicated in §3.8.5, Figure 3.11, then
\[ \hat{s} = \cos \theta \cdot \hat{n} + \sin \theta \cdot \hat{e}_\perp = \cos \theta \cdot \hat{n} + \sin \theta \cos \phi \cdot \hat{e}_1 + \sin \theta \sin \phi \cdot \hat{e}_2, \] (A.41)
where \( \hat{e}_\perp \) is the direction containing the tangent component of \( \hat{s} \) and \( \hat{e}_1, \hat{e}_2 \) are any couple of orthogonal versors spanning the tangent plane to the boundary and forming a right-handed triad with \( \hat{n} \). Defining
\[ \mathbb{S}_+^2 := \mathbb{S}^2 \cap \{ \hat{s} | \hat{n} \cdot \hat{s} > 0 \}, \]
the following results follow.

Proposition A.7. 
\[ \int_{\mathbb{S}_+^2} \hat{n} \cdot \hat{s} \, d\hat{s} = \pi. \] (A.42)

Proof. The result follows after the elementary integration
\[ \int_{\mathbb{S}_+^2} \hat{n} \cdot \hat{s} \, d\hat{s} = \int_{\mathbb{S}_+^2} \cos \theta \, d\hat{s} = \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta \int_0^{2\pi} d\phi = \pi. \]
\( \Box \)
Proposition A.8.
\[
\int_{S^2_+} \hat{s} \cdot \hat{n} \cdot \hat{s} \, d\hat{s} = \frac{2\pi}{3} \cdot \hat{n}.  \tag{A.43}
\]

Proof. From \( \hat{n} \cdot \hat{s} = \cos \theta \) and expanding \( \hat{s} \) according to (A.41),
\[
\int_{S^2_+} \hat{s} \cdot \hat{n} \cdot \hat{s} \, d\hat{s} = \hat{n} \cdot \int_{S^2_+} \cos^2 \theta \, d\hat{s} + \hat{e}_1 \cdot \int_{S^2_+} \cos \theta \sin \theta \cos \phi \, d\hat{s} + \hat{e}_2 \cdot \int_{S^2_+} \cos \theta \sin \theta \sin \phi \, d\hat{s}.
\]
The last two term on the r.h.s. vanish due to the presence of \( \cos \phi \) and \( \sin \phi \), to be integrated over a full period \([0, 2\pi]\). The only term left is integrated elementarily as
\[
\int_{S^2_+} \cos^2 \theta \, d\hat{s} = \int_0^{\pi/2} \cos^2 \theta \sin \theta \, d\theta \int_0^{2\pi} \, d\phi = \frac{2\pi}{3},
\]
whence the result. \(\square\)

### A.4 Stefan-Boltzmann Law from Planck Law

From the physical standpoint, Stefan-Boltzmann law trivially follows from Planck law by simply integrating over the full frequency spectrum. From the mathematical standpoint, the integration is not elementary. Even though a primitive function cannot be found (actually a nuisance when dealing with spectrum bands in non-gray radiation models), nonetheless a closed form result is possible for the whole spectrum, that is,
\[
\int_0^\infty \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1} \, d\nu = \frac{2\pi^4k^4}{15c^2h^3}T^4 = \frac{\sigma_{sb} T^4}{\pi},  \tag{A.44}
\]
where
\[
\sigma_{sb} := \frac{2\pi^5k^4}{15c^2h^3}
\]
is the Stefan-Boltzmann constant. After applying the change of variable \(u := h\nu/kT\), equality (A.44) follows from the following

Proposition A.9.
\[
\int_0^\infty \frac{u^3}{e^u - 1} \, du = \frac{\pi^4}{15}.  \tag{A.45}
\]

Proof. There is more than one way to prove this result. We like this argument, for its elegant number theoretic flavor, that applies to the more general case
\[
J(m) := \int_0^\infty \frac{u^m}{e^u - 1} \, du, \quad m \in \mathbb{N}.
\]
Since $1/e < 1$, the denominator may be expressed as the sum of the geometric series
\[
\sum_{n=0}^{\infty} (e^{-u})^n = \frac{1}{1 - e^{-u}},
\]
so that
\[
J(m) = \int_0^\infty u^m e^{-u} \sum_{n=0}^{\infty} e^{-nu} du = \int_0^\infty u^m \sum_{n=1}^{\infty} e^{-nu} du.
\]
Then, by the change of variables $t := nu$, one gets
\[
J(m) = \sum_{n=1}^{\infty} \frac{1}{n^{m+1}} \int_0^\infty t^m e^{-t} dt.
\]
The first factor on the r.h.s. is Riemann zeta function
\[
\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad z \in \mathbb{C},
\]
while the second factor is Euler Gamma function
\[
\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad z \in \mathbb{C},
\]
both evaluated for $z = m + 1 \in \mathbb{N} \subset \mathbb{C}$, so that
\[
J(m) = \zeta(m + 1) \Gamma(m + 1).
\]
In the case at hand, $m = 3$ and $J(3) = \zeta(4) \Gamma(4)$. In case of even integers, the Riemann zeta function may be notoriously evaluated by means of Bernoulli numbers $B_{2n}$ as [68]
\[
\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}, \quad n \in \mathbb{N}.
\]
All of Bernoulli numbers may be evaluated by solving the infinite sequence of linear equations
\[
\frac{1}{r!} \sum_{k=0}^{r-1} \binom{r}{k} B_k = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{if } r > 1 \end{cases}
\]
of which only the finite subset of the first ones is required to compute the finite subset of the first Bernoulli numbers [68]. Computationally more efficient methods also exists to compute Bernoulli numbers [50], but the level of complexity increases. In the case at hand, $n = 2$, $B_4 = -1/30$ and $\zeta(4) = \pi^4/90$. The portion with the Euler Gamma function is simple. Integrating by parts the property
\[
\Gamma(z) = z \cdot \Gamma(z - 1), \quad z \in \mathbb{C},
\]
is found. Then, since by direct integration $\Gamma(1) = 1$, by induction the celebrated property
\[
\Gamma(n) = (n-1)!, \quad n \in \mathbb{N} \subset \mathbb{C},
\]
is found, so that $\Gamma(4) = 3! = 6$ and $J(3) = \pi^4/15$. Summing up, (A.45) follows. \qed
Curricula Vitarum

**Andrea Balestrero** was born in Recco, Italy on July 13, 1980. He obtained a M.Sc. in Theoretical Physics, *cum laude*, from the Università degli Studi di Genova (University of Genoa, Italy) in 2004 with a thesis on quantum field theory over a non-commutative space-time. He joined ABB in 2005, where he is involved in the electric arc research and in the development of new simulation techniques for the interrupting process in low voltage breakers.

**Luca Ghezzi** was born in Gallarate, Italy on September 21, 1974. He obtained a M.Sc. in Structural Civil Engineering, *cum laude*, from the Politecnico di Milano (Technical University of Milan, Italy) in 1999 with a thesis on mathematical programming techniques applied to computational elasto-plasticity and poro-elasto-plasticity. He obtained a second M.Sc. in Mathematics, *cum laude*, from the Università degli Studi di Milano (University of Milan, Italy) in 2007 with a thesis on domain decomposition preconditioners for the Spectral Element Method applied to hyperbolic and elliptic PDE. He joined the Italian oil & gas company, ENI, in 2000 as a structure engineer and ABB Corporate Research in 2001, where he was involved in the development of computational methods for the virtual simulation of physical phenomena, including coupled, nonlinear problems such as arc plasma. In 2002 he moved to an ABB business unit manufacturing low voltage breakers, where he is currently responsible for a group carrying out mathematical modeling and numerical simulations.
# List of Symbols and Abbreviations

## Physical Constants

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Approximated value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>Speed of light (in vacuum)</td>
<td>$2.99792458 \cdot 10^8 \text{ m/s}$</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>Permittivity of free space</td>
<td>$8.854187817 \cdot 10^{-12} \text{ F/m}$</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>Permeability of free space</td>
<td>$1.256637061 \cdot 10^{-6} \text{ H/m}$</td>
</tr>
<tr>
<td>$k$</td>
<td>Boltzmann constant</td>
<td>$1.3806504 \cdot 10^{-23} \text{ J/K}$</td>
</tr>
<tr>
<td>$\sigma_{sb}$</td>
<td>Stefan-Boltzmann constant</td>
<td>$5.670400 \cdot 10^{-8} \text{ Wm}^{-2} \text{ K}^{-4}$</td>
</tr>
<tr>
<td>$h$</td>
<td>Planck constant</td>
<td>$6.62606896 \cdot 10^{-34} \text{ Js}$</td>
</tr>
<tr>
<td>$R$</td>
<td>(Universal) gas constant</td>
<td>$8.314472 \text{ JK}^{-1} \text{ mol}^{-1}$</td>
</tr>
<tr>
<td>$N_A$</td>
<td>Avogadro constant</td>
<td>$6.02214179 \cdot 10^{23} \text{ mol}^{-1}$</td>
</tr>
<tr>
<td>$u$</td>
<td>Unified atomic mass unit</td>
<td>$1.660538782 \cdot 10^{-27} \text{ kg}$</td>
</tr>
<tr>
<td>$e^-$</td>
<td>Electron charge</td>
<td>$1.60218 \cdot 10^{-19} \text{ C}$</td>
</tr>
<tr>
<td>$m_e$</td>
<td>Electron mass</td>
<td>$9.10938188 \cdot 10^{-31} \text{ kg}$</td>
</tr>
<tr>
<td>$G$</td>
<td>Gravitational constant</td>
<td>$6.67428 \cdot 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$</td>
</tr>
</tbody>
</table>
### Physical Quantities

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>Position</td>
</tr>
<tr>
<td>$\hat{s}$</td>
<td>Direction</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$V$</td>
<td>Volume</td>
</tr>
<tr>
<td>$f_s$</td>
<td>Phase-space density (for species $s$)</td>
</tr>
<tr>
<td>$n_s$</td>
<td>Number density (for species $s$)</td>
</tr>
<tr>
<td>$v$</td>
<td>Velocity (of a particle)</td>
</tr>
<tr>
<td>$u$</td>
<td>Average velocity (of a fluid)</td>
</tr>
<tr>
<td>$f$</td>
<td>Force</td>
</tr>
<tr>
<td>$W$</td>
<td>Work</td>
</tr>
<tr>
<td>$K$</td>
<td>Kinetic energy</td>
</tr>
<tr>
<td>$P$</td>
<td>Pressure tensor</td>
</tr>
<tr>
<td>$p$</td>
<td>(Hydrostatic) pressure</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>Pressure deviator</td>
</tr>
<tr>
<td>$T$</td>
<td>Temperature</td>
</tr>
<tr>
<td>$q$</td>
<td>Heat flux</td>
</tr>
<tr>
<td>$I_\nu$</td>
<td>Spectral radiative intensity</td>
</tr>
<tr>
<td>$G_\nu$</td>
<td>Total incident spectral radiation</td>
</tr>
<tr>
<td>$\alpha_\nu$</td>
<td>Absorptivity (or absorption coefficient)</td>
</tr>
<tr>
<td>$\sigma_\nu$</td>
<td>Scattering coefficient</td>
</tr>
<tr>
<td>$C_\nu$</td>
<td>Linear anisotropic phase function coefficient</td>
</tr>
<tr>
<td>$\Gamma_\nu$</td>
<td>Diffusion coefficient (for radiation)</td>
</tr>
<tr>
<td>$\epsilon_\nu$</td>
<td>Emissivity (or emission coefficient)</td>
</tr>
<tr>
<td>$U$</td>
<td>Specific internal energy</td>
</tr>
<tr>
<td>$h$</td>
<td>Specific enthalpy</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Total specific energy</td>
</tr>
<tr>
<td>$c_V$</td>
<td>Specific heat capacity at constant volume</td>
</tr>
<tr>
<td>$c_p$</td>
<td>Specific heat capacity at constant pressure</td>
</tr>
<tr>
<td>$E$</td>
<td>Electric field</td>
</tr>
<tr>
<td>$D$</td>
<td>Electric flux density</td>
</tr>
<tr>
<td>$B$</td>
<td>Magnetic flux density</td>
</tr>
<tr>
<td>$H$</td>
<td>Magnetic field</td>
</tr>
<tr>
<td>$q$</td>
<td>Charge</td>
</tr>
<tr>
<td>$\varrho$</td>
<td>Charge density (per unit volume)</td>
</tr>
<tr>
<td>$Z$</td>
<td>Charge number</td>
</tr>
<tr>
<td>$j$</td>
<td>Current density (per unit surface)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Electric conductivity</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Thermal conductivity</td>
</tr>
<tr>
<td>$m$</td>
<td>Mass</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Mass density (per unit volume)</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Electric permittivity</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Magnetic permeability</td>
</tr>
</tbody>
</table>
### Symbol Description

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_b )</td>
<td>Bulk viscosity</td>
</tr>
<tr>
<td>( \mu_d )</td>
<td>Dynamic viscosity</td>
</tr>
<tr>
<td>( \eta )</td>
<td>Magnetic diffusivity (or magnetic diffusivity)</td>
</tr>
<tr>
<td>( Re )</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>( Rm )</td>
<td>Magnetic Reynolds number</td>
</tr>
<tr>
<td>( i(t) )</td>
<td>Current time history</td>
</tr>
<tr>
<td>( u(t) )</td>
<td>Voltage time history</td>
</tr>
<tr>
<td>( g(t) )</td>
<td>Conductance time history</td>
</tr>
</tbody>
</table>

### Mathematical Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \simeq )</td>
<td>Isomorphism in between algebraic structures (particularly, vector space isomorphism)</td>
</tr>
<tr>
<td>( \sim )</td>
<td>Homotopy in between topological spaces or maps</td>
</tr>
<tr>
<td>( \approx )</td>
<td>Approximate equality</td>
</tr>
<tr>
<td>( ::= )</td>
<td>Equality by definition</td>
</tr>
<tr>
<td>( \equiv )</td>
<td>Identical equality</td>
</tr>
<tr>
<td>( \times )</td>
<td>Cross product (in Cartesian coordinates, ( (a \times b)_i = a_j b_k - a_k b_j ), for even permutations of ( i, j, k ))</td>
</tr>
<tr>
<td>( \circ )</td>
<td>Composition of maps, i.e., ( f \circ g = f(g) )</td>
</tr>
<tr>
<td>( \otimes )</td>
<td>Tensor product (e.g., ( (a \otimes b)_{ij} = a_i b_j ))</td>
</tr>
<tr>
<td>( \emptyset )</td>
<td>The empty set</td>
</tr>
<tr>
<td>( \cdot )</td>
<td>Dot product, or scalar product, or inner product in a Hilbert space ( (a \cdot b = \sum a_i b_i) )</td>
</tr>
<tr>
<td>( : )</td>
<td>Double dot product between rank 2 tensors ( (a : b = \sum \sum a_{ij} b_{ji}) )</td>
</tr>
<tr>
<td>( \partial )</td>
<td>Boundary of a manifold, when applied to a manifold, or exterior derivative, when applied to a differential form</td>
</tr>
<tr>
<td>( \partial_i )</td>
<td>Shorthand for ( \partial / \partial x_i )</td>
</tr>
<tr>
<td>( \partial_i^2 )</td>
<td>Shorthand for ( \partial^2 / \partial x_i \partial x_j )</td>
</tr>
<tr>
<td>( \partial_n )</td>
<td>Shorthand for ( \partial / \partial n )</td>
</tr>
<tr>
<td>( \nabla )</td>
<td>Gradient (in Cartesian coordinates, ( (\nabla a)_i = \partial_i a ))</td>
</tr>
<tr>
<td>( \nabla \cdot )</td>
<td>Divergence (in Cartesian coordinates, ( \nabla \cdot a = \sum \partial_i a_i ))</td>
</tr>
<tr>
<td>( \nabla \times )</td>
<td>Curl, or rotor (in Cartesian coordinates, ( (\nabla \times a)_i = \partial_j a_k - \partial_k a_j ))</td>
</tr>
<tr>
<td>( \nabla^2 )</td>
<td>Laplacian (( \nabla^2 = \nabla \cdot \nabla ))</td>
</tr>
<tr>
<td>( \mathbb{N} )</td>
<td>The set of natural numbers (without 0)</td>
</tr>
<tr>
<td>( \mathbb{N}_0 )</td>
<td>( \mathbb{N} \cup {0} )</td>
</tr>
<tr>
<td>( \mathbb{Z} )</td>
<td>The ring of integer numbers</td>
</tr>
<tr>
<td>( \mathbb{R} )</td>
<td>The real field</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\mathbb{R}^+$</td>
<td>The positive real semi-axis $[0, +\infty)$</td>
</tr>
<tr>
<td>$\mathbb{R}[x]$</td>
<td>The ring of polynomials over $\mathbb{R}$, in one variable $x$</td>
</tr>
<tr>
<td>$\mathbb{R}[x_1, \ldots, x_n]$</td>
<td>The ring of polynomials over $\mathbb{R}$, in $n$ variables $x_1, \ldots, x_n$</td>
</tr>
<tr>
<td>$S^1$</td>
<td>The unit circle, or 1-sphere (1-manifold)</td>
</tr>
<tr>
<td>$S^2$</td>
<td>The unit sphere, or 2-sphere (2-manifold)</td>
</tr>
<tr>
<td>$\mathbb{C}$</td>
<td>The complex field</td>
</tr>
<tr>
<td>$\text{Hom}(X, Y)$</td>
<td>The set of homomorphisms (i.e., linear maps) from vector space $X$ to vector space $Y$ (when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, then elements of this vector space may be identified with $m \times n$ matrices over $\mathbb{R}$)</td>
</tr>
<tr>
<td>$\text{End}(X)$</td>
<td>The monoid (under application composition) of endomorphisms of vector space $X$ in itself ($\text{End}(X) \equiv \text{Hom}(X, X)$ and when $X = \mathbb{R}^n$ then elements of this algebra may be identified with square matrices of order $n$)</td>
</tr>
<tr>
<td>$\text{Aut}(X)$</td>
<td>The group (under application composition) of automorphisms of vector space $X$ in itself (when $X = \mathbb{R}^n$ then elements of this algebra may be identified with non-singular, square, real matrices of order $n$)</td>
</tr>
<tr>
<td>$A^T$</td>
<td>Transpose matrix of the matrix $A$</td>
</tr>
<tr>
<td>$A^H$</td>
<td>Hermitian transpose matrix (also known as conjugate transpose matrix) of the matrix $A$</td>
</tr>
<tr>
<td>$A^\dagger$</td>
<td>Moore-Penrose pseudoinverse matrix (also known as generalized inverse matrix) of the matrix $A$</td>
</tr>
<tr>
<td>$TM$</td>
<td>The tangent bundle of a manifold $M$</td>
</tr>
<tr>
<td>$NM$</td>
<td>The normal bundle of a manifold $M$</td>
</tr>
<tr>
<td>$C^0(X)$</td>
<td>The vector space of continuous functions over space $X$</td>
</tr>
<tr>
<td>$C^1(X)$</td>
<td>The vector space of functions that are continuous over space $X$ together with their first derivatives</td>
</tr>
<tr>
<td>$C^2(X)$</td>
<td>The vector space of functions that are continuous over space $X$ together with their first and second derivatives</td>
</tr>
<tr>
<td>$C^k(X)$</td>
<td>The vector space of functions that are continuous over space $X$ together with their derivatives up to order $k$</td>
</tr>
<tr>
<td>$C^\infty(X)$</td>
<td>The vector space of functions that are continuous over space $X$ together with their derivatives of any order, i.e., the space of smooth functions on $X$</td>
</tr>
<tr>
<td>$L^1(X)$</td>
<td>The Lebesgue space of absolutely summable functions over space $X$, i.e., ${f : X \to \mathbb{C} \mid \int_X</td>
</tr>
<tr>
<td>$L^2(X)$</td>
<td>The Lebesgue space of square summable functions over space $X$, i.e., ${f : X \to \mathbb{C} \mid \int_X</td>
</tr>
<tr>
<td>$L^p(X)$</td>
<td>The Lebesgue space of functions whose $p$-th power is summable over space $X$, i.e., ${f : X \to \mathbb{C} \mid \int_X</td>
</tr>
<tr>
<td>$H^1(X)$</td>
<td>The Sobolev space of functions that are square summable over space $X$ together with their first derivatives (in distributional sense), i.e., ${f \in L^2(X) \mid \nabla f \in (L^2(X))^n}$, where $n = \dim X$</td>
</tr>
</tbody>
</table>
### Symbol Description

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{H}(\text{curl}; \ X)$</td>
<td>The space of vector fields which are square summable (component wise) over space $X$ together with their curl, i.e., ${ \mathbf{v} \in (L^2(\ X))^3 \mid \nabla \times \mathbf{v} \in (L^2(\ X))^3 }$</td>
</tr>
<tr>
<td>$|f|_2$</td>
<td>The $L^2$ norm of a function $f$ defined over space $X$, i.e., $\sqrt{\int_X</td>
</tr>
<tr>
<td>$|\mathbf{v}|_2$</td>
<td>The euclidean norm of a vector $\mathbf{v} \in \mathbb{C}^n$, i.e., $\sqrt{\sum_{i=1}^n</td>
</tr>
<tr>
<td>$|\circ|$</td>
<td>Shorthand for $|\circ|_2$</td>
</tr>
</tbody>
</table>

### Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>ABC</td>
<td>Absorbing Boundary Condition(s)</td>
</tr>
<tr>
<td>AC</td>
<td>Alternating Current</td>
</tr>
<tr>
<td>ACB</td>
<td>(Open) Air Circuit Breaker(s)</td>
</tr>
<tr>
<td>a.e.</td>
<td>almost everywhere (i.e., except on a set with measure zero in the sense of Lebesque measure theory)</td>
</tr>
<tr>
<td>AIS</td>
<td>Arc Imaging System</td>
</tr>
<tr>
<td>AMG</td>
<td>Algebraic MultiGrid</td>
</tr>
<tr>
<td>BC</td>
<td>Boundary Condition(s)</td>
</tr>
<tr>
<td>BDF</td>
<td>Backward Differentiation Formula(e)</td>
</tr>
<tr>
<td>CB</td>
<td>Circuit Breaker(s)</td>
</tr>
<tr>
<td>CFD</td>
<td>Computational Fluid Dynamics</td>
</tr>
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<td>CFE</td>
<td>Cold Field electron Emission</td>
</tr>
<tr>
<td>CP</td>
<td>Constricted Plasma</td>
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<td>CZ</td>
<td>Current Zero</td>
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<td>DAE</td>
<td>Differential Algebraic Equation(s)</td>
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<td>DC</td>
<td>Direct Current</td>
</tr>
<tr>
<td>DOM</td>
<td>Discrete Ordinate Method</td>
</tr>
<tr>
<td>FEM</td>
<td>Finite Element Method</td>
</tr>
<tr>
<td>FVM</td>
<td>Finite Volume Method</td>
</tr>
<tr>
<td>HID</td>
<td>High Intensity Discharge</td>
</tr>
<tr>
<td>HV</td>
<td>High Voltage</td>
</tr>
<tr>
<td>IC</td>
<td>Initial Condition(s)</td>
</tr>
<tr>
<td>iff</td>
<td>If and only if</td>
</tr>
<tr>
<td>IL</td>
<td>Ionization Layer</td>
</tr>
<tr>
<td>l.h.s.</td>
<td>left hand side</td>
</tr>
<tr>
<td>LTE</td>
<td>Local Thermal Equilibrium</td>
</tr>
<tr>
<td>LV</td>
<td>Low Voltage</td>
</tr>
<tr>
<td>MCB</td>
<td>Miniature Circuit Breaker(s)</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Description</td>
</tr>
<tr>
<td>--------------</td>
<td>--------------------------------------------</td>
</tr>
<tr>
<td>MCCB</td>
<td>Molded Case Circuit Breaker(s)</td>
</tr>
<tr>
<td>MHD</td>
<td>Magnetohydrodynamics</td>
</tr>
<tr>
<td>MV</td>
<td>Medium Voltage</td>
</tr>
<tr>
<td>NEC</td>
<td>Net Emission Coefficient</td>
</tr>
<tr>
<td>NRBC</td>
<td>Non Reflecting Boundary Condition(s)</td>
</tr>
<tr>
<td>ODE</td>
<td>Ordinary Differential Equation(s)</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial Differential Equation(s)</td>
</tr>
<tr>
<td>RTE</td>
<td>Radiative Transfer Equation</td>
</tr>
<tr>
<td>r.h.s.</td>
<td>right hand side</td>
</tr>
<tr>
<td>r.m.s.</td>
<td>root mean square</td>
</tr>
<tr>
<td>SI</td>
<td>Système International d’unités (International System of Units)</td>
</tr>
<tr>
<td>SH</td>
<td>Space charge sheath</td>
</tr>
<tr>
<td>SVD</td>
<td>Singular Value Decomposition</td>
</tr>
<tr>
<td>TF</td>
<td>Thermo-Field emission</td>
</tr>
<tr>
<td>TN</td>
<td>Layer of Thermal Non-equilibrium</td>
</tr>
<tr>
<td>TP</td>
<td>Layer of Thermal Perturbation</td>
</tr>
<tr>
<td>TRV</td>
<td>Transient Recovery Voltage</td>
</tr>
<tr>
<td>URF</td>
<td>Under-Relaxation Factor</td>
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<td>UV</td>
<td>Ultraviolet</td>
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