Coupled Fast Multipole Method - Finite Element Method for the analysis of magneto-mechanical problems

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Résumé

A numerical code coupling the Fast Multipole Method with both standard and non-standard Finite Element Methods is developed for the analysis of three dimensional magnetostatic-mechanical problems featuring infinite domains, non-linear ferromagnetic materials and large relative motions among different components

1 Introduction

The finite element method has obviously a dominant status in the field of computational methods in engineering, mostly because of its greater flexibility and wider range of applicability; on the contrary integral equations are superior for certain classes of problems featuring, in most cases, linear-elastic material behaviour, moving boundaries, infinite domains.

However, in certain cases, the optimal choice seems to be a coupled BEM-FEM approach. A model problem which evidences all the potential advantages of such coupling is depicted in Figure 1. A vertical conductor filament $A$ passes through a switch mechanism made of a fixed ferromagnetic lamina $B$ and a pivoting one $C$ (hinged on $B$). A sudden increase in current intensity through $A$ induces an attracting force between $B$ and $C$ (an elastic spring between $B$ and $C$ is calibrated to prevent motion for usual current intensities) which eventually induces a rotation of $C$. This mechanism actually constitutes a part of an industrial relay.

Laminae $B$ and $C$ are made of a ferromagnetic highly non-linear material and are hence discretized by means of domain methods.

The magnetic field in the air, on the contrary, is accounted for by the BEM, which essentially yields the following considerable advantages: (a) the infinite domain around the relay need not be discretized; (b) large rotations of the pivoting lamina $C$ can be easily accounted for, since no remeshing is required.
However, the development of such a numerical tool for the analysis of large scale problems turns out to be a challenging task for various reasons some of which are pointed out hereafter.

i) It is well known that classical FEM methods employing node-based shape functions experience severe difficulties in the modelling of general magnetic phenomena. A common remedy resorts to the so called wedge-elements ([2]); more advanced applications under investigation employ richer elements in the context of Discontinuous Local Galerkin approaches ([1]).

ii) Mesh adaptivity turns out to be often mandatory to achieve reasonable accuracy.

iii) Unfortunately the classical BEM for linear problems, both in the collocation and the variational versions, has essentially failed to live up to initial high expectations. Indeed the BEM produces fully populated matrix equations which make the application of direct solvers unrealistic for large scale problems; the computation of such matrices is numerically costly and hence iterative solvers cannot be adapted to the method as is. With respect to these key issues, recent investigations of the so called Fast Multipole Methods (FMM) seem to be changing the situation considerably. Indeed these methods, adapted to the BEM, allow to utilize iterative solvers (e.g. GMRES) and reduce the operation count per iteration to approximately $O(N \log^\alpha N)$ (to be compared with the $O(N^2)$ for classical approaches), where $\alpha$ is a positive number. The FMM can be considered as an efficient tool for evaluating the contribution to the integral equations stemming from regions which are far apart from each other, the “near field” contributions being evaluated by means of classical tools. FMM was
initially introduced by Rokhlin [10] as a technique for solving integral equation numerically, and later developed by Greengard as a fast solver for multibody (particle) interactions. In recent times one can observe a flourishing of different contributions and techniques ranging from the “panel clustering” method (e.g. [7]), to the “wavelet methods” (e.g. [8]) and finally to more efficient versions of the FMM (e.g. [5, 3]).

The results collected in this paper represent a first step of a medium term research program aiming at producing an effective numerical BEM-FFM-FEM tool collecting all the features previously described. Herein some simplifying assumptions are made, as detailed in the sequel.

2 Problem formulation

Let us assume that: i) current variation inside the conductor is slow enough to allow the use of a magnetostatic formulation; ii) the vertical filament $A$ is dealt with as a locus of a known current density distribution $J$.

The differential equations to be solved are as follows:

$$\text{curl} \, H_1 = J_s, \quad \text{div} \, B_1 = 0 \quad \text{in} \quad \Omega_1 \quad (1)$$

$$\text{curl} \, H_2 = J, \quad \text{div} \, B_2 = 0 \quad \text{in} \quad \Omega_2 \quad (2)$$

$$(H_1 - H_2) \wedge n = 0, \quad (B_1 - B_2) \cdot n = 0 \quad \text{on} \quad \partial \Omega \quad (3)$$

where: $H_i$ is the magnetic field intensity and $B_i$ is the magnetic flux density; $J_s$ accounts for self-induced currents in the ferromagnetic laminae $\Omega_1$; $\Omega_2$ represents the infinite domain surrounding the relay (and hence contains the filament $A$); $\partial \Omega$ is the interface between the two domains. The field variables satisfy the constitutive relations:

$$B_1 = \mu_1 (H_1) \quad \text{in} \quad \Omega_1, \quad B_2 = \mu_0 H_2 \quad \text{in} \quad \Omega_2$$

The numerical analysis of 3D magnetostatic problems is generally addressed by means of either scalar or vector potentials.

2.1 Scalar potential formulation.

Let us assume that $J_s = 0$ yielding curl$H_1 = 0$; hence a scalar potential $\phi_1$ exists such that $H_1 = \text{grad} \phi_1$ and div $(\mu_1 \text{grad} \phi_1) = 0$, the value of $\phi_1$ being fixed at some point of $\Omega_1$. The problem on $\Omega_1$ admits the equivalent variational formulation: find $\phi_1 \in H^1(\Omega_1)$ such that

$$\int_{\Omega_1} \text{grad} \phi_1(\mathbf{x}) \mu_1(\mathbf{x}) \text{grad} \phi_1(\mathbf{x}) \, d\Omega = \int_{\partial \Omega} \phi_1(\mathbf{x}) H_1(\mathbf{x}) \cdot n(\mathbf{x}) \, d\Gamma \quad \forall \phi_1 \in H^1(\Omega_1) \quad (4)$$

As for $\Omega_2$, let us set $H_2 = \text{grad} \phi_2 + H_a$ where $H_a$ accounts for $J$ and can be computed starting from the Biot-Savart law. The above problem is equivalent to the integral formulation:
find $\phi_2 \in H^{1/2}(\partial \Omega)$ such that
\[
k\phi_2(\mathbf{y}) = \int_{\partial \Omega} \left[ \mathbf{H}_2(\mathbf{x}) - \mathbf{H}_a(\mathbf{x}) \right] \cdot \mathbf{n}(\mathbf{x}) \, G(\mathbf{y}, \mathbf{x}) - \phi_2(\mathbf{x}) \left[ \text{grad} G(\mathbf{y}, \mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \right] \, d\Gamma \quad (5)
\]
where kernel $G(\mathbf{x}, \mathbf{y})$ is the classical potential theory Kelvin kernel:
\[
G = \frac{1}{4\pi} \frac{1}{\mathbf{r}}
\]
and $k$ depends on the geometry of $\partial \Omega$ at $y$. The equation set is completed by the continuity conditions on $\partial \Omega$:
\[
\phi_1 = \phi_2 + \int_{x_0}^{\mathbf{x}} \mathbf{H}_a \cdot \mathbf{t} \, d\ell \\
\mu_1 \mathbf{H}_1 \cdot \mathbf{n} = \mu_0 (\mathbf{H}_2 + \mathbf{H}_a) \cdot \mathbf{n} \quad (6)
\]
where $x_0$ is an arbitrary fixed point. It is worth stressing that the evaluation of the line integral (where $\mathbf{t}$ denotes the unit tangent vector and $\ell$ the arc length coordinate) is straightforward since “pseudo-potentials” based on Biot-Savart formula for $\mathbf{H}_a$ are available. Details will be provided in an extended version of the paper.

### 2.2 Vector potential formulation.

A more general formulation, not limited to the present assumption $\mathbf{J}_s = 0$, employs vector potentials. Since $\mathbf{B}_1$ is divergence free we can find $\mathbf{A}_1$ such that:
\[
\mathbf{B}_1 = \text{curl} \mathbf{A}_1
\]
In order to completely specify the vector potential $\mathbf{A}_1$ the so called Coulomb gauge is often adopted, imposing $\text{div} \mathbf{A}_1 = 0$. Eventually:
\[
\text{curl} \left( \frac{1}{\mu_1} \text{curl} \mathbf{A}_1 \right) = \mathbf{J}_s \quad \text{div} \mathbf{A}_1 = 0 \quad (7)
\]
which can be recast into the equivalent variational formulation: find $\mathbf{A}_1$ and $p$
\[
\mathbf{A}_1 \in H(\text{curl}, \Omega_1), \quad p \in H^1(\Omega_1)
\]
such that
\[
\int_{\Omega_1} \text{curl} \tilde{\mathbf{A}}_1 \cdot \frac{1}{\mu_1} \text{curl} \mathbf{A}_1 \, d\Omega - \int_{\Omega_1} \tilde{\mathbf{A}}_1 \cdot \text{grad} p \, d\Omega - \int_{\partial \Omega} \tilde{\mathbf{A}}_1 \cdot (\mathbf{H}_1 \wedge \mathbf{n}) \, d\Gamma + \int_{\Omega_1} \tilde{\mathbf{A}}_1 \cdot \mathbf{J}_s \, d\Omega \quad \forall \tilde{\mathbf{A}}_1 \in H(\text{curl}, \Omega_1)
\]
\[
- \int_{\Omega_1} \text{grad} \tilde{p} \cdot \mathbf{A}_1 \, d\Omega = 0 \quad \forall \tilde{p} \in H^1_0(\Omega_1) \quad (8)
\]
Non conventional “edge elements” are adopted in eqn. (8), since they guarantee inter-element continuity of the tangential component only, as required by the \( A_1 \in H(\text{curl}, \Omega_1) \) condition ([2]). The application of Discontinuous Galerkin schemes is currently under consideration.

Applying the same procedure to \( \Omega_2 \), and setting \( B_2 = \text{curl} A_2 \) one obtains :

\[
\begin{align*}
\mu_0 \text{curl}\text{curl} A_2 &= 0, \\
\text{div} A_2 &= 0 \\
\Rightarrow \\
\text{div}\text{grad} A_2 &= 0, \\
\text{div} A_2 &= 0
\end{align*}
\]  

(9)

The boundary condition \( B_1 \cdot n = B_2 \cdot n \) implies that the tangential derivatives of \( A_i \) is continuous across \( \partial \Omega \). Hence \( A_1 \) and \( A_2 \) on \( \partial \Omega \) can only differ by an arbitrary constant which will be henceforth set to zero. The integral equation stemming from the application of the third Green identity to \( \Omega_2 \) writes :

\[
k(y) A_2(y) = \int_{\partial \Omega} [\text{grad} A_2(x) \cdot n(x)] G(y, x) - A_2(x) [\text{grad} G(y, x) \cdot n(x)] \, d\Gamma
\]

\[
+ \frac{\mu_0}{4\pi} \int_{\Omega_A} \frac{J(x)}{r} \, d\Omega
\]

(10)

where \( \Omega_A \) denotes the filament \( A \). The condition \( \text{div} A_2 = 0 \) is automatically satisfied. Equation (10) can be transformed into:

\[
k(y) A_2(y) = \int_{\partial \Omega} \left( \mu_0 H_2(x) \wedge n(x) \right) G - [\text{grad} G(y, x) \cdot n(x)] A_2(x)
\]

\[
+ [\text{grad} G(y, x) \cdot n(x)] \cdot A_2(x) \right) \, d\Gamma + \frac{\mu_0}{4\pi} \int_{\Omega_A} \frac{J(x)}{r} \, d\Omega
\]

(11)

which is more suitable for coupling with eqn. (8). Equation (11) is collocated at every node of the interface \( \partial \Omega \).

The continuity relations :

\[
H_1 \wedge n = H_2 \wedge n, \quad A_1 = A_2 \quad \text{on} \quad \partial \Omega
\]

complete the set of governing equations.

### 3 Preliminary numerical results

A first set of encouraging numerical results was obtained for the relay of Figure 1. The formulation in Section 2.1 has been implemented with multipole accelerators for the BEM equation collocated at boundary nodes (details can be found in [5]). The isotropic non-linear
constitutive behaviour of the ferromagnetic laminae, expressing \( B_1 = \| B_1 \| \) as a function of \( H_1 = \| H_1 \| \), has been chosen as described in Figure 2. The two laminae have been meshed with linear tetraedrons, \( \partial \Omega \) with linear triangles. The non-symmetric linear system is solved employing a jacobi preconditioned GMRES routine. In Figure 2 the surface mesh is depicted; Figure 3 presents the results in terms of \( B_x \) surface field.

![Figure 2 – Mesh and constitutive law adopted for laminae B and C](image)

The auto-adaptive procedure is currently being implemented, together with the extension to the vector potential case.

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**Références**


Figure 3 – Map of $B_z$ for a current intensity of 1000 A


